OPTIMAL CONTROL PROBLEM FOR STOCHASTIC HIGHER ORDER SOBOLEV TYPE EQUATION

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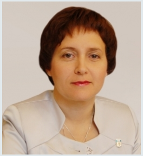

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- Introduction. Stating of the problem
- Stochastic Sobolev type equations of high order with relatively p-bounded operators (G. Sviridyuk, A. Zamyshlyaeva, O.Tzyplenkova)
  - The spaces of "noises". Stochastic K-processes. Phase space
  - Strong solutions
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- Multipoint initial-final value problems for dynamical Sobolev-type equations in the space of "noises"(G. Sviridyuk, S. Zagrebina)
- Linear Hoff equation with additive "white noise"(G. Sviridyuk, S. Zagrebina)
- Relatively radial operators in Hilbert spaces of differential k-forms with stochastic coefficients
- Stochastic Ginzburg – Landau equation

$$Av^{(n)} = Bv + f, \tag{1}$$

where operators A, B are linear and continuous, acting from Banach space $\mathfrak{V}$ to $\mathfrak{G}$, absolute term $f = f(t)$ models the external force.

$$A \overset{o}{\eta}{}^{(n)} = B\eta + w, \tag{2}$$

$$\overset{o}{\eta}{}^{(m)}(0) = \xi_m, \ m = 0, 1, \ldots, n-1, \tag{3}$$

$$A \overset{o}{\eta}{}^{(n)} = B\eta + w + Cu, \tag{4}$$

where $\eta = \eta(t)$ is a stochastic process, $\overset{o}{\eta}$ is the Nelson – Gliklikh derivative of process η, $w = w(t)$ is a stochastic process that responds for external influence; u is unknown control function from the Hilbert space $\mathfrak{U}$ of controls, operator $C \in \mathcal{L}(\mathfrak{U}; \mathfrak{G})$.

$$P\left(\overset{o}{\eta}{}^{(m)}(0) - \xi_m\right) = 0, \ m = 0, ..., n-1. \tag{5}$$

$(\hat{\eta}, \hat{u})$, where $\hat{\eta}$ is a solution to problem (4), (5), and the control $\hat{u}$ belongs to $\mathfrak{U}_{ad} \subset \mathfrak{U}$, and satisfies the relation

$$J(\hat{\eta}, \hat{u}) = \min_{(\eta,u)} J(\eta, u). \tag{6}$$

Here $J(\eta, u)$ is some specially constructed penalty functional and $\mathfrak{U}_{ad}$ is a closed convex set in the Hilbert space $\mathfrak{U}$ of controls.

$$d\eta = (S\eta + \psi)dt + Ad\omega, \tag{7}$$
$$L\mathring{\eta} = M\eta + N\omega, \tag{8}$$

1. Yu. E. Gliklikh; *Global and Stochastic Analysis with Applications to Mathematical Physics*, Springer, London, Dordrecht, Heidelberg, New York (2011). DOI: 10.1007/978-0-85729-163-9
2. M. Kovacs, S. Larsson; *Introduction to Stochastic Partial Differential Equations*, Proceedings of "New Directions in the Mathematical and Computer Sciences National Universities Commission, Abuja, Nigeria, October 8–12, 4 (2007), Lagos, Publications of the ICMCS, 159–232 (2008).
3. S. A. Zagrebina, E. A. Soldatova, G. A. Sviridyuk; *The Stochastic Linear Oskolkov Model of the Oil Transportation by the Pipeline, Semigroups of Operators – Theory and Applications*, Bedlewo, Springer International Publishing Switzerland, Heidelberg, New York, Dordrecht, London, 317-325 (2015). DOI: 10.1007/978-3-319-12145-1-20
4. A. Favini, G. A. Sviridyuk, A. A. Zamyshlyaeva; *One class of Sobolev type equation of higher order with additive "white noise"*, Communications on Pure and Applied Analysis, American Institute of Mathematical Sciences, 15 (1), 185–196 (2016). DOI: 10.3934/cpaa.2016.15.185
5. A. Favini, G. A. Sviridyuk, N. A. Manakova; *Linear Sobolev Type Equations with Relatively p-Sectorial Operators in Space of "noises"*, Abstract and Applied Analysis, Hindawi Publishing Corporation, 2015 (2015). DOI: 10.1155/2015/697410
6. G. A. Sviridyuk, V. E. Fedorov; *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*. Utrecht, Boston, Köln, Tokyo, VSP, 2003. DOI: 10.1515/9783110915501
7. A. Favini, A. Yagi; *Degenerate differential equations in Banach spaces*. New York, Basel, Hong Kong, Marcel Dekker, Inc., 1999.
8. G. V. Demidenko, S. V. Uspenskii; *Partial differential equations and systems not solvable with respect to the highest – order derivative*. New York, Basel, Hong Kong, Marcel Dekker, Inc., 2003.

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be the complete probability space. A measurable mapping $\xi : \Omega \to \mathbb{R}$ is called *a random variable*. Let $\mathcal{A}_0$ be a σ-subalgebra of σ-algebra $\mathcal{A}$. Construct subspace $\mathbf{L_2^0} \subset \mathbf{L_2}$ of random variables measurable with respect to $\mathcal{A}_0$. Denote the orthoprojector by $\Pi : \mathbf{L_2} \to \mathbf{L_2^0}$. Let $\xi \in \mathbf{L_2}$, then $\Pi\xi$ is called *a conditional expectation* of the random variable ξ, and is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$.
Consider a set $\mathfrak{I} \subset \mathbb{R}$ and the following two mappings. $f : \mathfrak{I} \to \mathbf{L_2}$, associates to each $t \in \mathfrak{I}$ the random variable $\xi \in \mathbf{L_2}$. $g : \mathbf{L_2} \times \Omega \to \mathbb{R}$, associates to each pair (ξ, ω) the point $\xi(\omega) \in \mathbb{R}$. The mapping $\eta : \mathbb{R} \times \Omega \to \mathbb{R}$ having form $\eta = \eta(t, \omega) = g(f(t), \omega)$ is called a *stochastic process*. The stochastic process $\eta = \eta(t, \cdot)$, where f and g are detined above, is a random variable for each fixed $t \in \mathfrak{I}$, i.e. $\eta(t, \cdot) \in \mathbf{L_2}$, and $\eta = \eta(\cdot, \omega)$ is called *a (sample) path* for each fixed $\omega \in \Omega$. The stochastic process η is called *continuous*, if all its paths are almost sure continuous (i.e. for almost all $\omega \in \Omega$ the paths $\eta(\cdot, \omega)$ are continuous). The set of continuous stochastic processes forms a Banach space, which is denoted by $\mathbf{C}(\mathfrak{I}, \mathbf{L_2})$. Fix $\eta \in \mathbf{C}(\mathfrak{I}, \mathbf{L_2})$ and $t \in \mathfrak{I}$, and denote by $\mathcal{N}_t^\eta$ the σ-algebra generated by the random variable $\eta(t)$. For brevity, $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$.

Definition

Let $\eta \in \mathbf{C}(\mathfrak{I}, \mathbf{L_2})$. A random process

$$\overset{o}{\eta} = \frac{1}{2}\left(\lim_{\Delta t \to 0+} \mathsf{E}_t^\eta\left(\frac{\eta(t+\Delta t, \cdot) - \eta(t, \cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \mathsf{E}_t^\eta\left(\frac{\eta(t, \cdot) - \eta(t-\Delta t, \cdot)}{\Delta t}\right)\right)$$

is called a *Nelson–Gliklikh derivative* $\overset{o}{\eta}$ *of the stochastic process* η *at point* $t \in \mathfrak{I}$, if the limits exist in the sense of the uniform metric on $\mathbb{R}$.

The spaces $\mathbf{C}^l(\mathfrak{I}, L_2)$ are called *the spaces of differentiable "noises"*. Let $\mathfrak{I} = \{0\} \cup \mathbb{R}_+$, then a well–known example of a vector in the space $\mathbf{C}^l(\mathfrak{I}, L_2)$ is given by a stochastic process that describes the Brownian motion in Einstein–Smoluchowski model

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin \frac{\pi}{2}(2k+1)t,$$

where the independent random variables $\xi_k \in \mathbf{L}_2$ are such that the variances $D\xi_k = [\frac{\pi}{2}(2k+1)]^{-2}$, $k \in \{0\} \cup \mathbb{N}$.

$$\overset{o}{\beta}(t) = \frac{\beta(t)}{2t}, t \in \mathbb{R}_+.$$

Now let $\mathfrak{U}$ be a real separable Hilbert space with orthonormal basis $\{\varphi_k\}$. Denote by $\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2$ the Hilbert space, which is a completion of the linear span of *random variables*

$$\eta = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_k\varphi_k, \quad \|\eta\|^2_{\mathfrak{U}} = \sum_{k=1}^{\infty} \lambda_k \mathbf{D}\xi_k.$$

The sequence $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}_+$ is such that $\sum\limits_{k=1}^{\infty} \lambda_k < +\infty$, $\{\xi_k\} \subset \mathbf{L}_2$ is a sequence of random variables. The elements of $\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2$ will be called random $\mathbf{K}$-variables.

Mapping $\eta : (\varepsilon, \tau) \to \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2$ given by

$$\eta(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_k(t)\varphi_k,$$

where the sequence $\{\xi_k\} \subset \mathbf{C}(\mathfrak{I}, L_2)$, is called a $\mathfrak{V}$-*valued continuous stochastic* $\mathbf{K}$-*process*, if the series on the right-hand side converges uniformly on any compact in $\mathfrak{I}$ by norm $\|\cdot\|_{\mathfrak{V}}$, and paths of process $\eta = \eta(t)$ are almost sure continuous. Continuous stochastic $\mathbf{K}$-process $\eta = \eta(t)$ is called *continuously Nelson–Gliklikh differentiable on* $\mathfrak{I}$, if the series

$$\overset{o}{\eta}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \overset{o}{\xi}_k(t)\varphi_k$$

converges uniformly on any compact in $\mathfrak{I}$ in the norm $\|\cdot\|_{\mathfrak{V}}$, and paths of process $\overset{o}{\eta}=\overset{o}{\eta}(t)$ are almost sure continuous. A stochastic $\mathbf{K}$-process, which is continuously differentiable up to any order $l \in \mathbb{N}$ inclusively, is a Wiener $\mathbf{K}$-process

$$W_{\mathbf{K}}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)\varphi_k,$$

where $\{\beta_k\} \subset \mathbf{C}^l(\mathfrak{I}, L_2)$ is a sequence of Brownian motions on $\mathbb{R}_+$. Similarly, if G is a real separable Hilbert space with orthonormal basis $\{\varphi_k\}$, the spaces $\mathbf{C}(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ and $\mathbf{C}^l(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$, $l \in \mathbb{N}$, are constructed. Note also that spaces $\mathbf{C}^l(\mathfrak{I}, L_2)$, $\mathbf{C}(\mathfrak{I}, \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2)$ and $\mathbf{C}^l(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$, $l \in \mathbb{N}$, are called *the spaces of differentiable* $\mathbf{K}$-*"noises"* .

$A, B \in \mathcal{L}(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}, \mathfrak{G}_{\mathbf{K}}\mathbf{L_2})$. $\rho^A(B) = \{\mu \in \mathbb{C} : (\mu A - B)^{-1} \in \mathcal{L}(\mathfrak{G}_{\mathbf{K}}\mathbf{L_2}, \mathfrak{V}_{\mathbf{K}}\mathbf{L_2})\}$ and $\sigma^A(B) = \mathbb{C} \setminus \rho^A(B)$. $(\mu A - B)^{-1}$, $R^A_\mu(B) = (\mu A - B)^{-1}A$, $L^A_\mu(B) = A(\mu A - B)^{-1}$ If the set $\sigma^A(B)$ is bounded ($\exists a > 0 :\ (|\mu| < a) \Rightarrow \mu \in \sigma^A(B)$) then the operator B is called (A, σ)-bounded.
Let the operator B be (A, σ)-bounded, $p \in \{0\} \cup \mathbb{N}$.
$\sigma^A_n(B) = \{\mu \in \mathbb{C} : \mu^n \in \sigma^A(B)\}$; $\gamma = \{\mu \in \mathbb{C} : |\mu| = r, r^n > a\}$

$$P = \frac{1}{2\pi i}\int\limits_\gamma \mu^{n-1} R^A_{\mu^n}(B) d\mu \in \mathcal{L}(\mathfrak{V}_{\mathbf{K}}\mathbf{L}_2), \quad Q = \frac{1}{2\pi i}\int\limits_\gamma \mu^{n-1} L^A_{\mu^n}(B) d\mu \in \mathcal{L}(\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2).$$

Here, $R^A_{\mu^n}(B) = (\mu^n A - B)^{-1}A$ and $L^A_{\mu^n}(B) = A(\mu A - B)^{-1}$. Put $\mathfrak{V}^0_{\mathbf{K}}\mathbf{L}_2(\mathfrak{V}^1_{\mathbf{K}}\mathbf{L}_2) = \ker P(\mathrm{im}\ P)$, $\mathfrak{G}^0_{\mathbf{K}}\mathbf{L}_2(\mathfrak{G}^1_{\mathbf{K}}\mathbf{L}_2) = \ker Q(\mathrm{im}\ Q)$. Thus, the spaces $\mathfrak{V}_{\mathbf{K}}\mathbf{L}_2$ and $\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2$ since P and Q are projectors, can be decomposed into direct sums $\mathfrak{V}_{\mathbf{K}}\mathbf{L}_2 = \mathfrak{V}^0_{\mathbf{K}}\mathbf{L}_2 \bigoplus \mathfrak{V}^1_{\mathbf{K}}\mathbf{L}_2$ and $\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2 = \mathfrak{G}^0_{\mathbf{K}}\mathbf{L}_2 \bigoplus \mathfrak{G}^1_{\mathbf{K}}\mathbf{L}_2$, whereas $\mathfrak{V}^0_{\mathbf{K}}\mathbf{L}_2 \supset \ker A$. By $A_k(B_k)$ define the restriction of operator $A(B)$ onto $\mathfrak{V}^k_{\mathbf{K}}\mathbf{L_2}$, $k = 0, 1$.

Lemma

The operators $A_k, B_k \in \mathcal{L}(\mathfrak{V}^k_{\mathbf{K}}\mathbf{L_2}; \mathfrak{G}^k_{\mathbf{K}}\mathbf{L_2}), k = 0, 1$; moreover, there exist the operators $B_0^{-1} \in \mathcal{L}(\mathfrak{G}^0_{\mathbf{K}}\mathbf{L_2}; \mathfrak{V}^0_{\mathbf{K}}\mathbf{L_2})$ and $A_1^{-1} \in \mathcal{L}(\mathfrak{G}^1_{\mathbf{K}}\mathbf{L_2}; \mathfrak{V}^1_{\mathbf{K}}\mathbf{L_2})$.

$H = B_0^{-1}A_0 \in \mathcal{L}(\mathfrak{V}^0_{\mathbf{K}}\mathbf{L_2})$, $S = A_1^{-1}B_1 \in \mathcal{L}(\mathfrak{V}^1_{\mathbf{K}}\mathbf{L_2})$.

The (A, σ)-bounded operator B is called *(A, p)-bounded*, $p \in \{0\} \cup \mathbb{N}$, if ∞ is a removable singular point (i.e. $H \equiv \mathbb{O}, p = 0$) or the pole of order $p \in \mathbb{N}$ (i.e. $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$) of the *$A$-resolvent* $(\mu A - B)^{-1}$ of operator B.

$$A \overset{o}{\eta}{}^{(m)}(0) = A\xi_m, \ m = 0, ..., n-1,$$

and has advantages over the Cauchy condition (3) in the case of Sobolev type equations

$$\lim_{t \to 0+} P\left(\overset{o}{\eta}{}^{(m)}(t) - \xi_m\right) = 0, \ m = 0, ..., n-1. \tag{9}$$

The K-random process $\eta \in \mathbf{C}^n(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ is called *a classical solution of equation* (2), if a.s. all its trajectories satisfy equation (2) for some K-random process $w \in \mathbf{C}(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$. The solution $\eta = \eta(t)$ of (2) is called *the classical solution* of problem (2), (9) if a.s. condition (9) is also fulfilled. The classical solutions of the problems (2), (5) and (2), (3) are defined analogously. Consider firstly problem (3) for the homogeneous equation

$$A \overset{o}{\eta}{}^{(n)} = B\eta. \tag{10}$$

In this case (and only in this case) consider $\mathfrak{I} = \mathbb{R}$.

Definition

The mapping $V \in C^\infty(\mathbb{R}; \mathcal{L}(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}))$ is called *a propagator* of equation (10), if for all $v \in \mathfrak{V}_{\mathbf{K}}\mathbf{L_2}$ the vector-function $\eta(t) = V(t)v$ is a solution of (10).

Theorem

Let the operator B be (A,σ)-bounded. Then, the operator-functions

$$V_m(t) = \frac{1}{2\pi i}\int_\gamma \mu^{n-m-1}(\mu^n A - B)Ae^{\mu t}d\mu,$$

where $m = 0, 1, \ldots, n-1$ and the integral is understood in the sense of Riemann, define the propagators of equation (10).

Lemma

$V_m^\bullet \in C^\infty(\mathbb{R}; \mathcal{L}(\mathfrak{V}_\mathbf{K}\mathbf{L_2}; \mathfrak{V}^1_\mathbf{K}\mathbf{L_2}))$, $(V_m(t))^{(l)}_t = V_{m+l}(t)$, *where* $m = 0, 1, \ldots, n-1$, $l = 0, 1, \ldots, m$; $(V_m(t))^{(l)}_t\Big|_{t=0} = \mathbb{O}$ *for* $m \neq l$ *and* $(V_m(t))^{(m)}_t\Big|_{t=0} = P$ *is the projector in* $\mathfrak{V}_\mathbf{K}\mathbf{L_2}$ *on* $\mathfrak{V}^1_\mathbf{K}\mathbf{L_2}$ *along* $\mathfrak{V}^0_\mathbf{K}\mathbf{L_2}$.

Definition

The set $\mathfrak{P} \subset \mathfrak{V}_\mathbf{K}\mathbf{L_2}$ is called *the phase space* of equation (10) if
(i) a.s. every trajectory of the solution $\eta = \eta(t)$ lies in $\mathfrak{P}$ pointwise, i.e. $\eta(t) \in \mathfrak{P}$ a.s. for all $t \in \mathbb{R}$;
(ii) for all random variables $\xi_m \in L_2(\Omega; \mathfrak{P})$, $m = 0, 1, \ldots, n-1$, there exists a unique solution $\eta \in \mathbf{C}^n_K\mathbf{L}_2$ of (3), (10).

Theorem

Let the operator B be (A,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then the subspace $\mathfrak{V}^1_{\mathbf{K}}\mathbf{L_2}$ is the phase space of equation (10).

Corollary

Under the conditions of Theorem the solution of (3), (10) is the Gaussian K-random process if the random variables $\xi_m,\ m = 0, 1, \dots, n-1$, are Gaussian.

Lemma

Let the operator B be (A,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then for all independent random variables $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2,\ m = 0, 1, \dots, n-1$, there exists a.s. a unique solution $\eta \in \mathbf{C}^{\infty}_{K}\mathbf{L}_2$ of (5), (10), represented in the form $\eta(t) = \sum\limits_{m=0}^{n-1} V^t_m \xi_m$, $t \in \mathbb{R}$. If in addition $\xi_m,\ m = 0, 1, \dots, n-1$ take values only in $\mathfrak{V}^1_{\mathbf{K}}\mathbf{L_2}$, then this solution is a unique solution of (3), (10).

Let the K-random process $w = w(t), t \in [0, \tau)$ be such that

$$(\mathbb{I} - Q)w \in \mathbf{C}^{n(p+1)}(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2) \text{ and } Qw \in \mathbf{C}(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2), \tag{11}$$

then the K-random process

$$\eta(t) = -\sum_{q=0}^{p} H^q B_0^{-1}(\mathbb{I} - Q)N \overset{o}{w}^{(qn)}(t) + \int_0^t V_{n-1}^{t-s} A_1^{-1} QNw(s)ds \tag{12}$$

is a unique classical solution of (5), (2) with $\xi_m \in \mathfrak{V}^0_{\mathbf{K}}\mathbf{L_2}, \ m = 0, ..., n-1.$

Theorem

Let the operator B be (A, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. For any K-random process $w = w(t)$ satisfying (11), and for all independent random variables $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2, \ m = 0, 1, \ldots, n-1$, independent with w, there exists a.s. a unique solution $\eta \in \mathbf{C}^n(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ of (2), (5), represented in the form

$$\eta(t) = \sum_{m=0}^{n-1} V_m^t \xi_m - \sum_{q=0}^{p} H^q B_0^{-1}(\mathbb{I} - Q) \overset{o}{w}^{(qn)}(t) + \int_0^t V_{n-1}^{t-s} A_1^{-1} Qw(s)ds. \tag{13}$$

Definition

A vector function

$$\eta \in H^n(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}) = \{\eta \in L_2(\mathfrak{I}; \mathfrak{V}_{\mathbf{K}}\mathbf{L_2}) : \overset{o}{\eta}^{(n)} \in L_2(\mathfrak{I}; \mathfrak{V}_{\mathbf{K}}\mathbf{L_2})\}$$

is called a *strong solution* of equation (2), if it a.s. turns the equation to identity almost everywhere on interval $(0, \tau)$. A strong solution $\eta = \eta(t)$ of equation (2) is called a *strong solution to problem* (2), (5) if condition (5) a.s. holds.

$H^n(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}) \hookrightarrow C^{n-1}(\mathfrak{I}; \mathfrak{V}_{\mathbf{K}}\mathbf{L_2})$.

$$H^{np+n}(\mathfrak{G}_{\mathbf{K}}\mathbf{L_2}) = \{v \in L_2(\mathfrak{I}; \mathfrak{G}_{\mathbf{K}}\mathbf{L_2}) : \overset{o}{v}^{(np+n)} \in L_2(\mathfrak{I}; \mathfrak{G}_{\mathbf{K}}\mathbf{L_2}), p \in \{0\} \cup \mathbb{N}\}.$$

Let $w \in H^{np+n}(\mathfrak{G}_{\mathbf{K}}\mathbf{L_2})$. Introduce the operators

$$A_1 w(t) = -\sum_{q=0}^{p} H^q B_0^{-1}(\mathbb{I} - Q) \overset{o}{w}^{(qn)}(t),$$
$$A_2 w(t) = \int_0^t V_{n-1}^{t-s} A_1^{-1} Q w(s) ds, t \in (0, \tau)$$

and the function

$$k(t) = \sum_{m=0}^{n-1} V_m^t \xi_m.$$

Lemma

Let the operator B be (A,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then
(i) $A_1 \in \mathcal{L}(H^{np+n}(\mathfrak{G}_{\mathbf{K}}\mathbf{L_2}); H^n(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}))$;
(ii) for arbitrary $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2,\ m = \overline{0, n-1}$ *the vector function* $k \in C^n([0,\tau); \mathfrak{V}_{\mathbf{K}}\mathbf{L_2})$;
(iii) $A_2 \in \mathcal{L}(H^{np+n}(\mathfrak{G}_{\mathbf{K}}\mathbf{L_2}); H^n(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}))$.

$$A \overset{o}{\eta}{}^{(n)} = B\eta + w, \tag{2}$$

$$P\left(\overset{o}{\eta}{}^{(m)}(0) - \xi_m\right) = 0,\ m = 0, ..., n-1. \tag{5}$$

Theorem

Let the operator B be (A,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. For any K-random process $w = w(t)$ satisfying (11), and for all independent random variables $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2,\ m = 0, 1, \dots, n-1$, independent with w, there exists a.s. a unique strong solution to problem (2), (5).

$$\overset{o}{H}{}^{np+n}(\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2) = \{u \in L_2(0,\tau;\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2) : u^{(np+n)} \in L_2(0,\tau;\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2), u^{(q)}(0) = 0 \ a.s., q = \overline{0,p}\},$$

$p \in \{0\} \cup \mathbb{N}$. In the space $\overset{o}{H}{}^{np+n}(\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2)$ we single out a closed convex subset $\overset{o}{H}{}_{\partial}^{np+n}(\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2)$, which will be called *the set of admissible controls.*

Definition

A vector function $\hat{u} \in \overset{o}{H}{}_{\partial}^{np+n}(\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2)$ is called an *optimal control of solutions to problem* (4), (5), if relation (6) holds.

We need to prove the existence of a unique control $\hat{u} \in \overset{o}{H}{}_{\partial}^{np+n}(\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2)$, minimizing the penalty functional

$$J(\eta,u) = \mu \sum_{q=0}^{n} \int_0^{\tau} || \overset{o}{\eta}{}^{(q)} - \overset{o}{\tilde{\eta}}{}^{(q)} ||^2_{\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2} dt + \nu \sum_{q=0}^{np+n} \int_0^{\tau} \left\langle N_q \overset{o}{u}{}^{(q)}, \overset{o}{u}{}^{(q)} \right\rangle_{\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2} dt. \qquad (14)$$

Here $\mu, \nu > 0$, $\mu + \nu = 1$, $N_q \in \mathcal{L}(\mathfrak{U}_{\mathbf{K}}\mathbf{L}_2)$, $q = 0, 1, \ldots, np+n$, are self-adjoint positively defined operators, and $\tilde{\eta}(t)$ is the target state of the system.

Theorem

Let the operator B be (A,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then for arbitrary $w \in H^{np+n}(\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ there exists a unique optimal control to solutions of problem (4), (5).

Introduce the following condition:

$$\left.\begin{array}{c} \sigma^L(M) = \bigcup\limits_{j=0}^{m} \sigma_j^L(M), \text{ for } m \in \mathbb{N}; \text{ moreover }, \sigma_j^L(M) \neq \emptyset, \text{ there exist closed} \\ \text{contours } \gamma_j \subset \mathbb{C}, \text{ bounding corresponding domains } D_j \supset \sigma_j^L(M), \text{ such that} \\ \overline{D_j} \cap \sigma_0^L(M) = \emptyset \text{ and } \overline{D_k} \cap \overline{D_l} = \emptyset \text{ for all } j,k,l = \overline{1,m} \text{ with } k \neq l. \end{array}\right\} \quad (A1)$$

Consider the linear stochastic Sobolev-type equation

$$L\overset{\circ}{\eta} = M\eta + N\omega, \tag{15}$$

where $\eta = \eta(t)$ is the required stochastic K-process and $\omega = \omega(t)$ is a known stochastic K-process, and the operator $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$.
Take $\tau_0 = 0$ and $\tau_j \in \mathbb{R}_+$ with $\tau_{j-1} < \tau_j$ for $j = \overline{1,m}$. Complement (15) with the multipoint initial-final conditions

$$\lim_{t\to\tau_0+} P_0(\eta(t) - \xi_0) = 0, \quad P_j(\eta(\tau_j) - \xi_j) = 0, \quad j = \overline{1,m}. \tag{16}$$

$$\xi_j = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_{jk}\varphi_k, \; j = \overline{0,m}, \tag{17}$$

where $\xi_{jk} \in \mathbf{L}_2$ is a Gaussian random variable such that series (17) is convergent. Call a stochastic K-process $\eta \in \mathbf{C}_K^1\mathbf{L}_2$ a (*classical*) *solution* to (15) whenever a.s. all its trajectories satisfy (15) for some stochastic K-process $\omega \in \mathbf{C}_K\mathbf{L}_2$, some operator $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, and all $t \in \mathcal{I}$. Call a solution $\eta = \eta(t)$ to (15) a (*classical*) *solution* to problem (15), (16) whenever in addition condition (16) is satisfied.

Theorem

For $p \in \{0\} \cup \mathbb{N}$ *take an* (L,p)*-bounded operator* M *and assume that condition (A1) holds. Given* $\tau_j \in \mathbb{R}_+$ *for* $j = \overline{1,m}$, *an operator* $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, *a nuclear operator* $K \in \mathcal{L}(\mathfrak{U})$ *with real spectrum* $\sigma(K)$, *a stochastic* K*-process* $\omega = \omega(t)$ *such that* $(\mathbb{I} - Q)N\omega \in \mathbf{C}_K^{p+1}\mathbf{L}_2$ *and* $QN\omega \in \mathbf{C}_K\mathbf{L}_2$, *and random variables* $\xi_j \in \mathbf{L}_2$, *for* $j = \overline{0,m}$, *such that* (17) *are fulfilled, there exists a unique solution* $\eta \in \mathbf{C}_K^1\mathbf{L}_2$ *to problem* (15), (16); *moreover, it is of the form*

$$\eta(t) = -\sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I} - Q)\overset{\circ}{\omega}^{(q)}(t) + \sum_{j=0}^{m}\left[U_j^{t-\tau_j}\xi_j + \int_{\tau_j}^{t} U_j^{t-\tau_j-s} L_{1j}^{-1} Q_j N\omega(s)ds\right], \quad t \in \mathcal{I}. \tag{18}$$

Corollary

If all the hypotheses of Theorem 16 hold and $\omega(t) = \overset{\circ}{W}_K(t)$ *then, given random variables* $\xi_j \in \mathbf{L}_2$ *as in* (17), *there exists a unique solution to problem* (15), (16); *furthermore, it has the form*

$$\eta(t) = \sum_{j=0}^{m}\left[U_j^{t-\tau_j}\xi_j - S_j P_j \int_{\tau_j}^{t} U_j^{t-\tau_j-s} L_{1j}^{-1} Q_j N W_K(s)ds + L_{1j}^{-1} Q_j N W_K(t)\right] - \sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I} - Q)\,\overset{\circ}{W}{}_K^{(q+1)}(t), \quad t \in \overline{\mathbb{R}}_+. \tag{19}$$

Consider a bounded domain $D \subset \mathbb{R}^d$ $(d \in \mathbb{N})$ with boundary ∂D of class C^∞. Denote $\mathfrak{U} = \{u \in W_2^{l+2}(D) : u(x) = 0, x \in \partial D\}$ and $\mathfrak{F} = W_2^l(D)$, where $l \in \{0\} \cup \mathbb{N}$.
Fixing $\alpha, \mu \in \mathbb{R}$, construct the operators $L = \mu\mathbb{I} + \Delta$ and $M = \alpha\mathbb{I}$, where Δ is the Laplace operator. Consider also the spectral problem

$$-\Delta u = \nu u \text{ in } D \text{ and } u(x) = 0 \text{ for } x \in \partial D. \tag{20}$$

Its solution is a family $\{\nu_j\} \subset \mathbb{R}_+$ of eigenvalues enumerated in the nondecreasing order taking their multiplicities into account and accumulating only to $+\infty$, as well as the associated orthonormal (in the sense of $\mathcal{F}$) family of eigenfunctions $\{\varphi_j\}$.
For all $\mu \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ the operator M is $(L, 0)$-bounded; moreover, its L-spectrum is

$$\sigma^L(M) = \{\mu_k = \frac{\alpha}{\mu - \nu_k},\ k \in \mathbb{N} \setminus \{l : \mu = \nu_l\}\} \cup \{0\}. \tag{21}$$

Furthermore, for $m \in \mathbb{N}$ construct the operator $\Lambda = (-\Delta)^m$ with

$$\operatorname{dom} \Lambda = \{u \in W_2^{l+2m}(D) : \Delta^k u(x) = 0,\ x \in \partial D,\ k = \overline{0, m-1}\}.$$

The family of eigenfunctions of Λ coincides with the family $\{\varphi_j\}$, while its family of eigenvalues is $\{\nu_j^m\}$. Since their asymptotics is $\nu_j^m \sim j^{\frac{2m}{d}} \to \infty$ as $j \to \infty$, we can choose $m \in \mathbb{N}$ so that, firstly, the dimension d of the domain D has some acceptable physical meaning, and secondly, the series $\sum_{j=1}^{\infty} (\nu_j^m)^{-1}$ converges.

Then the Green operator of Λ is nuclear, and we take it as K. Therefore, consider the linear stochastic Hoff equation in the form

$$L\overset{\circ}{\eta} = M\eta + \overset{\circ}{W}_K, \tag{22}$$

where L and M are defined above, while N is the embedding operator $\mathbb{I} : \mathfrak{U} \hookrightarrow \mathfrak{F}$ and $\overset{\circ}{W}_K = \overset{\circ}{W}_K(t)$ is the Nelson-Gliklikh derivative of the $\mathfrak{U}$-valued Wiener K-process $W_K = W_K(t)$, for $t \in \mathbb{R}_+$.

Take the projectors

$$P(Q) = \begin{cases} \mathbb{I}_{\mathfrak{U}}(\mathbb{I}_{\mathfrak{F}}) \text{ if } \mu \neq \nu_j \ \forall j \in \mathbb{N}; \\ \mathbb{I}_{\mathfrak{U}} - \sum_{j:\mu=\nu_j} \langle \cdot, \varphi_j \rangle \varphi_j \left(\mathbb{I}_{\mathfrak{F}} - \sum_{j:\mu=\nu_j} \langle \cdot, \psi_j \rangle \psi_j \right), \end{cases}$$

Furthermore, choose $h \in \mathbb{R}_+$ with $h < \max_{j\in\mathbb{N}}\{|\nu_j|\}$ and construct the projectors

$$\begin{gathered} P_1 = \mathbb{I}_{\mathfrak{U}} - \sum_{h<|\nu_j|} \langle \cdot, \varphi_j \rangle_{\mathfrak{U}} \varphi_j, \quad Q_1 = \mathbb{I}_{\mathfrak{F}} - \sum_{h<|\nu_j|} \langle \cdot, \psi_j \rangle_{\mathfrak{F}} \psi_j; \\ P_0 = P - P_1, \quad Q_0 = Q - Q_1. \end{gathered} \tag{23}$$

Observe that in the construction of these projectors condition (A1) holds because $\sigma_0^L(M) = \{\mu_j \in \sigma^L(M) : |\nu_j| \leq h\}$ and $\sigma_1^L(M) = \{\mu_j \in \sigma^L(M) : |\nu_j| > h\}$; hence, $\sigma_0^L(M) \cap \sigma_1^L(M) = \emptyset$.

Choose $\tau_1 \in \mathbb{R}_+$ as well as random variables ξ_0 and ξ_1 independent of each other and of stochastic K-processes η and pose the initial-final conditions

$$\lim_{t\to 0+} P_0(\eta(t) - \xi_0) = 0, \quad P_1(\eta(\tau_1) - \xi_1) = 0, \tag{24}$$

$$\xi_0 = \sum_{k=1}^{\infty} \sqrt{\nu_k}\xi_{0k}\varphi_k, \quad \xi_1 = \sum_{k=1}^{\infty} \sqrt{\nu_k}\xi_{1k}\varphi_k. \tag{25}$$

Theorem

If condition (A1) *is satisfied then for all numbers* $\mu \in \mathbb{R}$, $\alpha \in \mathbb{R}\setminus\{0\}$ *and* $\tau_1 \in \mathbb{R}_+$, *as well as random variables* ξ_{0k} *and* ξ_{1k} *such as* $\mathbf{D}\xi_{0k} \le C_0$ *and* $\mathbf{D}\xi_{1k} \le C_1$ *for some* $C_0, C_1 \in \mathbb{R}_+$ *there exists a unique solution* $\eta = \eta(t)$, *for* $t \in \mathbb{R}_+$, *to problem* (22), (24); *furthermore, it is of the form*

$$\begin{aligned}\eta(t) = {} & (L_{10}^{-1}Q_0 + L_{11}^{-1}Q_1)W_K(t) - L_{11}^{-1}Q_1W_K(\tau_1) \\ & - S_0P_0\int_0^t U_0^{t-s}L_{10}^{-1}Q_0W_K(s)ds + U_0^t\xi_0 + U_1^{t-\tau_1}\xi_1 \\ & - S_1P_1\int_{\tau_1}^t U_1^{t-\tau_1-s}L_{11}^{-1}Q_1W_K(s)ds - M_0^{-1}(\mathbb{I}-Q)N\overset{\circ}{W}_K(t),\end{aligned} \tag{26}$$

for $t \in \mathbb{R}_+$.

Here

$$
\begin{gathered}
U_0^t = \sum_{\nu_j \in \sigma_0^L(M)} e^{t\mu_j} \langle \cdot, \varphi_j \rangle_{\mathcal{U}} \varphi_j, \quad U_1^t = \sum_{\nu_j \in \sigma_1^L(M)} e^{t\mu_j} \langle \cdot, \varphi_j \rangle_{\mathcal{U}} \varphi_j, \\
L_{10}^{-1} = \sum_{\nu_j \in \sigma_0^L(M)} (\mu - \nu_j)^{-1} \langle \cdot, \varphi_j \rangle_{\mathcal{U}} \varphi_j, \\
L_{11}^{-1} = \sum_{\nu_j \in \sigma_1^L(M)} (\mu - \nu_j)^{-1} \langle \cdot, \varphi_j \rangle_{\mathcal{U}} \varphi_j, \\
S_{10} = \alpha \sum_{\nu_j \in \sigma_0^L(M)} (\mu - \nu_j)^{-1} \langle \cdot, \varphi_j \rangle_{\mathcal{U}} \varphi_j, \\
S_{11} = \alpha \sum_{\nu_j \in \sigma_1^L(M)} (\mu - \nu_j)^{-1} \langle \cdot, \varphi_j \rangle_{\mathcal{U}} \varphi_j, \\
M_0^{-1} = \alpha^{-1} \sum_{\nu_j = \mu} \langle \cdot, \psi_j \rangle_{\mathcal{F}} \psi_j.
\end{gathered}
\tag{27}
$$

Definition

The operator M is said to be *p-radial with respect to an operator L*(or *(L,p)-radial*), if it satisfies the following two conditions:
(i) $\exists\ a \in \mathbb{R}\ \forall\ \mu > a\ \mu \in \rho^L(M)$;
(ii) $\exists\ K > 0\ \forall\ \mu_k > a,\ k = 0, ..., p,\ \forall n \in \mathbb{N}$

$$\max\left\{\left\|\left(R^L_{(\mu,p)}(M)\right)^n\right\|_{\mathcal{L}(\mathbf{U_K L_2})}, \left\|\left(L^L_{(\mu,p)}(M)\right)^n\right\|_{\mathcal{L}(\mathbf{F_K L_2})}\right\} \leq \frac{K}{\prod\limits_{k=0}^{p}(\mu_k - a)^n}.$$

Definition

A mapping $V^\bullet \in C(\mathbb{R}_+; \mathcal{L}(\mathbf{H_K L_2}))$ is called a *semigroup* in a Hilbert space $\mathbf{H_K L_2}$, if

$$V^s V^t = V^{s+t} \quad \forall s, t \in \mathbb{R}_+.$$

Let us identify the semigroup with its graph $\{V^t : t \in \mathbb{R}_+\}$. The semigroup $\{V^t : t \in \mathbb{R}_+\}$ will be called a *C_0-semigroup (strongly continuous semigroup)*, if it is strongly continuous for $t > 0$ and there exists $\lim\limits_{t\to 0+} V^t v = v$ a.s. (i.e., for almost all $\omega \in \Omega$). The set
$\ker V^\bullet = \{v \in \mathbf{H_K L_2} :\ \ \mathtt{a.s.}\, V^t v = 0\ \exists t \in \mathbb{R}_+\}$ will be called the *kernel*, and the set
$\mathrm{im} V^\bullet = \{v \in \mathbf{H_K L_2} :\ \ \mathtt{a.s.}\, v = V^0 v\}$ – the *image* of the semigroup $\{V^t : t \in \mathbb{R}_+\}$.

Theorem (14)

Let M be an (L,p)-radial operator. Then there exists a C_0-semigroup of operators on the space $\mathbf{U_K L}_2$ ($\mathbf{F_K L}_2$).

$$U^t = s - \lim_{k\to\infty}\left(\frac{k(p+1)}{t} R^L_{\frac{k(p+1)}{t}}(M)\right)^{k(p+1)} \in \mathcal{L}(\mathbf{U_K L}_2) \tag{28}$$

$$\left(F^t = s - \lim_{k\to\infty}\left(\frac{k(p+1)}{t} L^L_{\frac{k(p+1)}{t}}(M)\right)^{k(p+1)} \in \mathcal{L}(\mathbf{F_K L}_2)\right).$$

Denote $\ker U^\bullet = \mathbf{U}^0_\mathbf{K}\mathbf{L}_2$, $\ker F^\bullet = \mathbf{F}^0_\mathbf{K}\mathbf{L}_2$, $\mathrm{im} U^\bullet = \mathbf{U}^1_\mathbf{K}\mathbf{L}_2$, $\mathrm{im} F^\bullet = \mathbf{U}^1_\mathbf{K}\mathbf{L}_2$, and by $L^k(M^k)$ denote the restriction of the operator $L(M)$ to $\mathbf{U}^k_\mathbf{K}\mathbf{L}_2$ $(\mathrm{dom} M \bigcap \mathbf{U}^k_\mathbf{K}\mathbf{L}_2)$ for $k = 0, 1$.

Lemma

Let M be an (L,p)-radial operator. Then the following assertions hold:
(i) $L_0 \in \mathcal{L}(\mathbf{U}^0_\mathbf{K}\mathbf{L}_2; \mathbf{F}^0_\mathbf{K}\mathbf{L}_2$ and $M_0 \in Cl(\mathbf{U}^0_\mathbf{K}\mathbf{L}_2; \mathbf{F}^0_\mathbf{K}\mathbf{L}_2)$;
(ii) There exists an operator $M_0^{-1} \in \mathcal{L}(\mathbf{F}^0_\mathbf{K}\mathbf{L}_2; \mathbf{U}^0_\mathbf{K}\mathbf{L}_2)$;
(iii) The operator $H = M_0^{-1}L$ $\left(G = LM_0^{-1}\right)$ is nilpotent with degree $\leq p$.

Assume that there exists an operator

$$L_1^{-1} \in \mathcal{L}(\mathbf{F}^1_{\mathbf{K}}\mathbf{L}_2; \mathbf{U}^1_{\mathbf{K}}\mathbf{L}_2) \tag{29}$$

and that the spaces $\mathbf{U}_{\mathbf{K}}\mathbf{L}_2$ and $\mathbf{F}_{\mathbf{K}}\mathbf{L}_2$ split as follows:

$$\mathbf{U}_{\mathbf{K}}\mathbf{L}_2 = \mathbf{U}^0_{\mathbf{K}}\mathbf{L}_2 \oplus \mathbf{U}^1_{\mathbf{K}}\mathbf{L}_2, \mathbf{F}_{\mathbf{K}}\mathbf{L}_2 = \mathbf{F}^0_{\mathbf{K}}\mathbf{L}_2 \oplus \mathbf{F}^1_{\mathbf{K}}\mathbf{L}_2. \tag{30}$$

Remark

Conditions (29) and (30) are satisfied if the Hilbert spaces $\mathbf{U}_{\mathbf{K}}\mathbf{L}_2$ andd $\mathbf{F}_{\mathbf{K}}\mathbf{L}_2$ are reflexive or if the operator M is strongly (L, p)-radial.

Lemma

Let M be an (L, p)-radial operator, and let conditions (29) and (30) be satisfied. Then $L_1 \in \mathcal{L}(\mathbf{U}^1_{\mathbf{K}}\mathbf{L}_2; \mathbf{F}^1_{\mathbf{K}}\mathbf{L}_2)$ and $M_0 \in Cl(\mathbf{U}^0_{\mathbf{K}}\mathbf{L}_2; \mathbf{F}^0_{\mathbf{K}}\mathbf{L}_2)$.

Any of the splittings (30) of a space is equivalent to the existence of the corresponding projector. This projector has the form $s - \lim\limits_{t \to 0+} U^t$.

Theorem

Let M be an (L, p)-radial operator. Then the operator $S = L_1^{-1} M_1$ $\left(T = M_1 L_1^{-1}\right)$ is the generator of the C_0-semigroup $U_1^\bullet$ $(F_1^\bullet)$ that is the restriction of the semigroup $U^\bullet$ $(F^\bullet)$ to the space $\mathbf{U}^1_{\mathbf{K}}\mathbf{L}_2$ $\left(\mathbf{F}^1_{\mathbf{K}}\mathbf{L}_2\right)$.

Consider the linear stochastic Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta. \tag{31}$$

A stochastic $\mathbf{K}$-process $\eta \in \mathbf{C}^1(\mathfrak{I}; \mathbf{U_K L}_2)$ will be called a (*solution of (31)* if substituting it into this equation a.s. results in an identity.

Definition

A set $\mathfrak{P} \subset \mathbf{U_K L}_2$ will be called *the phase space* of (31) if its satisfied the following conditions:
(i) Each trajectory of the solution $\eta = \eta(t)$ to (31) a.s. lies in $\mathfrak{P}$;
(ii) For a.a. $\eta_0 \in \mathfrak{P}$ there exists a solution of (31) with the condition $\eta(0) = \eta_0$.

Theorem

Let M be an (L, p)-radial operator, and let conditions (29) and (30)be satisfied. Then the phase space of (31) coincides with the image of the resolving semigroup of the form (28).

Consider the inhomogeneous equation

$$L \overset{o}{\eta} = M\eta + \omega, \tag{32}$$

where the vector function ω belongs to the space $C^\infty(\mathfrak{I}; \mathbf{F_K L}_2)$, $\mathfrak{I} = [0, t)$. Let M be an (L, p)-radial operator, and let conditions (29) and (30) be satisfied. Then (32) can be considered in the form of the system of two equations

$$H\dot{\eta}^0 = \eta^0 + M_0^{-1}(\mathbb{I} - Q)\omega^0, \tag{33}$$

$$\dot{\eta}^1 = S\eta^1 + L_1^{-1}Q\omega^1.$$

Corollary

By Lemma 4, the operator H is nilpotent; therefore, the Cauchy problem $\eta^0(0) = \eta_0^0$ for (33) is unsolvable for

$$\eta_0^0 \neq -\sum_{q=0}^{p} H^p M_0^{-1} \frac{d^q \omega^0}{dt^q}(0).$$

Consequently, for the Cauchy problem $\eta(0) = \eta_0$ for (32) to be uniquely solvable, it is necessary to impose auxiliary conditions depending on the right-hand side of the equation on the vector η_0.

By the preceding, for the initial conditions we consider the Showalter–Sidorov conditions

$$\lim_{t\to 0+}(R^L_\alpha(M))^{p+1}(\eta(L)-\eta_0)=0. \tag{34}$$

Let M be an (L,p)-radial operator, and let conditions (29), (30), be satisfied; then relation (34) is equivalent to the condition

$$P(\eta(0)-\eta_0)=0.$$

Theorem

Let M be an (L,p)-radial operator, let conditions (29), (30) be satisfied, and let the inclusion $\omega\in C^\infty(\mathfrak{J};\mathbf{F_K L}_2)$ hold, $\mathfrak{J}=[0,T)$. Then for each $\eta_0\in\mathbf{U_K L}_2$ there exists a solution $\eta\in C^1(\mathfrak{J};\mathbf{U_K L}_2)$ of the Showalter–Sidorov problem (34) for (32), and it has the form

$$\eta(t)=U^t\eta_0-\sum_{q=0}^{p}H^pM_0^{-1}\frac{d^q\omega^0}{dt^q}(t)+\int\limits_0^t U^{t-s}L_1^{-1}Q\omega ds.$$

Let $\mathcal{M}$ be a smooth compact oriented Riemannian manifold without boundary with local coordinates $x_1, x_2, ..., x_n$. By $H_k = H_k(\mathcal{M}, \Omega)$ denote the space of smooth differential k-forms $k = 0, 1, 2, ..., n$ with stochastic coefficients of the form

$$\chi_{i_1,i_2,\ldots,i_k}(t, x_1, x_2, ..., x_n, \omega) = \sum_{|i_1,i_2,\ldots,i_k|=k} a_{i_1,i_2,\ldots,i_k}(t, x_{i_1}, x_{i_2}, ..., x_{i_k}, \omega) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k},$$

where $a_{i_1,i_2,\ldots,i_k}(t, x_{i_1}, x_{i_2}, ..., x_{i_k}, \omega)$ are coefficients depending, among other variables, on time, and $|i_1, i_2, ..., i_k|$ is a multi-index.
One has the standard inner product

$$(\xi, \varepsilon)_0 = \int_{\mathcal{M}} \xi \wedge *\varepsilon, \quad \xi, \varepsilon \in H_k. \tag{35}$$

Here $*$ is the Hodge operator and $\wedge$ is the operator of exterior multiplication of k-forms. Completing the space H_k by continuity in the norm $\| \cdot |_0$ corresponding to the inner product (35) we obtain the space $\mathfrak{H}_k^0$. Introducing inner product in the spaces of differentiable or twice differentiable (in the Nelson–Gliklikh sense) k-forms and completing the space in the norms corresponding to these inner products, we construct the spaces $\mathfrak{H}_k^1$ and $\mathfrak{H}_k^2$. For these Hilbert spaces, one has continuous embeddings

$$\mathfrak{H}_k^2 \subseteq \mathfrak{H}_k^1 \subseteq \mathfrak{H}_k^0.$$

In the spaces constructed, we can use a Laplace – Beltrami operator

$$\Delta u = d\delta + \delta du,$$

where d is the operator of exterior differentiation of differential forms and the operator $\delta = *d*$ is the adjoint of the operator d.
The following generalization of the Hodge–Kodaira theorem holds for the resulting spaces.

Theorem

For the space $\mathfrak{H}^l_k, l = 0, 1, 2,$ one has the following decomposition into the direct sum of subspaces:

$$\mathfrak{H}^l_k = \mathfrak{H}^l_{kd} \oplus \mathfrak{H}^l_{k\delta} \oplus \mathfrak{H}^l_{k\Delta}, l = 0, 1, 2,$$

where $\mathfrak{H}_{kd}$ is the space of potential forms, $\mathfrak{H}_{k\delta}$ is the space of solenoidal forms, and $\mathfrak{H}_{kd}$ is the space of harmonic forms.

Let $\mathbf{K} = \{\lambda_k\}$ be a sequence such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. By $\{\varphi_k\}$ and $\{\psi_k\}$ denote the systems of eigenvectors of the Laplace–Beltrami operator orthonormal with respect to the inner products in $\mathfrak{H}_k^0$ and $\mathfrak{H}_k^2$. These systems form bases in the spaces $\mathfrak{H}_k^0$ and $\mathfrak{H}_k^2$. The elements of the spaces $\mathfrak{H}_{k\mathbf{K}}^0\mathbf{L}_2$ and $\mathfrak{H}_{k\mathbf{K}}^2\mathbf{L}_2$ are vectors $\chi = \sum\limits_{k=1}^{\infty} \lambda_k \xi_k \varphi_k$ and $\kappa = \sum\limits_{k=1}^{\infty} \lambda_k \zeta_k \psi_k$, the sequences of random variables $\{\xi_k\} \subset \mathbf{L}_2$ and $\{\zeta_k\} \subset \mathbf{L}_2$ are such that $\mathbf{D}\xi_k \leq const$ and $\mathbf{D}\zeta_k \leq const$. Construct the set of continuous processes $\mathbf{C}(\mathfrak{I}; \mathfrak{H}_{k\mathbf{K}}^0\mathbf{L}_2)$ and the set of continuously Nelson – Gliklikh differentiable processes $\mathbf{C}^1(\mathfrak{I}; \mathbf{H}_{k\mathbf{K}}^0\mathbf{L}_2)$.
Define operators $L,\ M : \mathbf{H}_{k\mathbf{K}}^2\mathbf{L}_2 \to \mathbf{H}_{k\mathbf{K}}^0\mathbf{L}_2$ by the formulas

$$L = \lambda + \Delta,\ M = \nu\Delta - id\Delta^2 \tag{36}$$

and reduce (32) to the equation

$$L \overset{\circ}{\chi} = M\chi. \tag{37}$$

Lemma

For any $\nu,\ \lambda,\ d \in \mathbb{R}$, the operator M is strongly $(L, 0)$-radial.

Theorem

(i) If $\lambda \notin \{\sigma_k\}$, then the phase space of (37) coincides with the space $\mathfrak{H}_{k\mathbf{K}}^0\mathbf{L}_2$.
(ii) If $\lambda \in \{\sigma_k\}$, then the phase space of (37) is the space
$\mathcal{P} = \{\varepsilon \in \mathbf{H}_{k\mathbf{K}}^0\mathbf{L}_2 :< \varepsilon, \varphi_l >= 0, \sigma_l = \lambda\}$.

Consider the inhomogeneous stochastic Ginzburg – Landau equation

$$(\lambda + \Delta)\chi_t = \nu\Delta\chi - id\Delta^2\chi + \theta \tag{38}$$

in the space of differential forms with stochastic coefficients $\mathfrak{H}^q_{0\mathbf{K}}\mathbf{L}_2$ given on smooth compact oriented Riemannian manifolds without boundary. Making the change of variables by formula (15) and denoting the inhomogeneity by $\omega = \Theta$, we obtain an equation of the form (32) and can apply abstract theorem to the problem with the Showalter–Sidorov condition

$$P(\chi(0) - \chi_0) = 0. \tag{39}$$

Theorem

For any $\nu,\ \lambda,\ d \in \mathbb{R}$ *any vector function* $\omega \in C^\infty(\mathfrak{I};\mathbf{F_K L}_2)$, $\mathfrak{I} = [0,t)$, *and an arbitrary* $\chi_0 \in \mathbf{U_K L}_2$ *there exists a solution* $\chi \in C^1(\mathfrak{I};\mathbf{U_K L}_2)$ *of the Showalter–Sidorov problem (39) for (38), which has the form*

$$\chi(t) = U^t\chi_0 - \sum_{q=0}^{p}(L_1^{-1}M_1)^p M_0^{-1}\frac{d^q\omega^0}{dt^q}(t) + \int\limits_0^t U^{t-s}\omega ds,$$

where $U^t = \begin{cases} \sum\limits_{k=1}^{+\infty} e^{\mu_k t} < \cdot, \varphi_k > \varphi_k, \\ \sum\limits_{k:\sigma_k \neq \lambda} e^{\mu_k t} < \cdot, \varphi_k > \varphi_k, \end{cases}$ *and the operators are defined by the relations*

$$L_1^{-1} = \begin{cases} \sum\limits_{k=1}^{+\infty} (\lambda + \sigma_k)^{-1} < \cdot, \varphi_k > \varphi_k, \\ \sum\limits_{k:\sigma_k \neq \lambda} (\lambda + \sigma_k)^{-1} < \cdot, \varphi_k > \varphi_k, \end{cases} \quad M_1 = \begin{cases} \sum\limits_{k=1}^{+\infty} (\nu\sigma_k - id\sigma_k^2) < \cdot, \varphi_k > \varphi_k, \\ \sum\limits_{k:\sigma_k \neq \lambda} (\nu\sigma_k - id\sigma_k^2) < \cdot, \varphi_k > \varphi_k, \end{cases}$$

$$M_0^{-1} = \begin{cases} \mathbb{O}, \ \sigma_k \neq \lambda, \ \forall k \in \mathbb{N}, \\ \sum\limits_{k:\sigma_k \neq \lambda} (\nu\sigma_k - id\sigma_k^2)^{-1} < \cdot, \varphi_k > \varphi_k. \end{cases}$$

Thank you for your attention!