On periodic BRW on $\mathbb{Z}^d$ with heavy tails

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A random walk

Suppose there is a particle at a point $v \in \mathbb{Z}^d$ at some time t. This particle can either go to another point $u \in \mathbb{Z}^d$ or remain at v for a short period of time δt.

a probability of the transition $v \to u$

$$p(v, u, \delta t) = a(v, u)\delta t + o(\delta t),$$

a probability of the transition $v \to v$ (i.e. particle remains in v)

$$p(v, v, \delta t) = 1 + a(v, v)\delta t + o(\delta t).$$

The transition intensity

The value $a(v,u)$ is called the transition intensity between v and u.

(i)

$$a(v,u) \geqslant 0, \quad v \neq u;$$

(ii)

$$a(v,v) < 0;$$

(iii)

$$\sum_{u \in \mathbb{Z}^d} a(v,u) = 0;$$

Let $g_1, \dots, g_d$ be a basis of $\mathbb{R}^d$ with integer coordinates. We define a lattice Γ by:

$$\Gamma = \left\{ g \in \mathbb{Z}^d : g = \sum_{j=1}^{d} n_j g_j,\ \ n_j \in \mathbb{Z}, j = 1, \dots, d \right\}.$$

(iv)

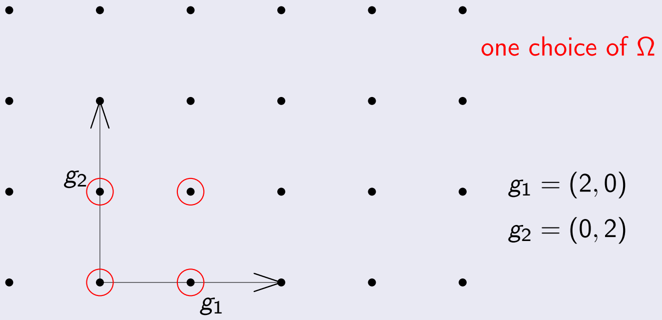
$a(v,u) = a(u,v) = a(v+g, u+g), \quad \forall g \in \Gamma;$

Ω

We can choose a set of vertices $\Omega = \{v_1, \dots, v_p\}$ such that for any $u \in \mathbb{Z}^d$ there is an unique representation

$$u = \omega_u + \gamma_u, \quad \omega_u \in \Omega, \gamma_u \in \Gamma.$$

Example



$g_1 = (2, 0)$

$g_2 = (0, 2)$

The transition intensity

Let $x = \{x_1, x_2, \dots, x_d\} \in \mathbb{R}^d$. We denote by $\|x\|_\infty$ the following norm

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \dots |x_d|\}.$$

Let P be a transition matrix from the standard basis $\{e_1, \dots, e_d\}$ to the basis $\{g_1, \dots, g_d\}$

$$g_j = Pe_j.$$

(v)

There is $\alpha \in (0, 2)$ such that

$$a(v_j + g, v_k)\|P^{-1}g\|_\infty^{d+\alpha} \to h_{jk} \text{ for } \|g\| \to +\infty,$$

where $h_{jk} \in [0, +\infty)$, for all $j, k = 1, \dots, p$ and at least for one pair j, k corresponding h_{jk} is strictly positive.

The transition intensities

(vi)

For any $v, u \in \Omega$ and $g \in \Gamma$

$$a(v, u - g) = a(v, u + g).$$

(vii)

The graph $G = (\mathbb{Z}^d, \mathcal{E})$ with the vertex set $\mathbb{Z}^d$ and edge set

$$\mathcal{E} = \{(v, u) : \ a(v, u) > 0, \ \ v, u \in \mathbb{Z}^d\}$$

is connected.

A branching source

Suppose that a particle located in a vertex $v \in \mathbb{Z}^d$ with branching source can't move anywhere from there. We assume that in a short period of time δt the particle can generate several offsprings.

a probability of generating $k \neq 1$ offsprings

$$p_k(v) = b_k(v)\delta t + o(\delta t).$$

a probability of generating $k = 1$ offsprings ($\equiv$ nothing happens)

$$p_1(v) = 1 + b_1(v)\delta t + o(\delta t).$$

The branching sources

(1)

$b_k \geqslant 0, \quad k \neq 1;$

(2)

$b_1 \leqslant 0;$

(3)

$$\sum_{k=0}^{+\infty} b_k = 0;$$

(4)

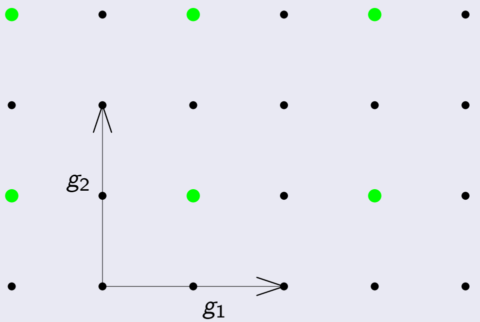
$$\beta = \sum_{k=1}^{+\infty} k b_k < \infty;$$

Locations of the branching sources

(5)

$$\beta(v+g) = \beta(v), \quad g \in \Gamma.$$

Example



$$g_1 = (2,0)$$
$$g_2 = (0,2)$$

A random walk with periodic set of branching sources

We assume that all particles evolve independently of each other. Eventually, a particle located at the point $v \in \mathbb{Z}^d$ at time t for the short period of time δt can jump to another point $u \in \mathbb{Z}^d$, decay into $k \neq 1$ offsprings, or remain still.

a probability of transition $v \to u$

$$p(v, u, \delta t) = a(v, u)\delta t + o(\delta t),$$

a probability of generating $k \neq 1$ offsprings in v

$$p_k(v, \delta t) = b_k(v)\delta t + o(\delta t).$$

a probability that nothing happens

$$p(v, \delta t) = 1 + a(v, v)\delta t + b_k(v)\delta t + o(\delta t).$$

The mean value of particles

Suppose that there is a particle at $t = 0$ at a point v. Denote by $M(v, u, t)$ the mean value of the number of particles at a time t at a point u. The function $M(v, u, t)$ satisfies the following Cauchy problem:

The differential equation

$$\begin{cases} \partial_t M(v,u,t) = \mathcal{A} M(v,u,t), \\ M(v,u,0) = \delta_u(v). \end{cases} \tag{1}$$

The operator $\mathcal{A}$

$$\mathcal{A} = \mathcal{A}_0 + Q,$$

$$(\mathcal{A}_0 f)(v) = \sum_{w \in \mathbb{Z}^d} a(v,w) f(w),$$

$$(Qf)(v) = \beta(v) f(v).$$

A discrete Laplacian

Sunada T., Sy P. [1992]

Local spectral properties near the left edge of the spectrum.

Higuchi Y., Shirai T. [2001,2004]

Direct integral decomposition. Spectral properties. Local spectral properties near the left edge for the magnetic Shrödinger.

Higuchi Y., Nomura Y. [2008]

Direct integral decomposition. Conditions for absolute continuity of the spectrum (absence of flat bands).

Korotyaev E., et. all [2010 – 2018]

Direct integral decomposition with explicit form of fibre operators. Estimates for spectrum of Laplacian with different types of perturbations.

Yarovaya E., et. all [1998-2007]

One source, finite variance. Asymptotic behaviour of all moments. Limit theorems.

Yarovaya E. [2013]

BRW with heavy tails, $d = 1, 2$, limit theorems.

Rytova A, Yarovaya E.[2016, 2019]

the multidimensional analog of the lemma, asymptotic behaviour of moments.

Khristolyubov I., Yarovaya E. [2019]

N sources. Asymptotic behaviour of all moments in subcritical and supercritical cases. Limit theorems.

Main result

Theorem 1

Let a BRW satisfy conditions (i – vii) and $(a - e)$. Then the following asymptotic relation holds for large t

$$M(v, u, t) = m(v, u)e^{t \sup \sigma(A)} t^{-\frac{d}{\alpha}}(1 + o(1)),$$

where $\alpha \in (0, 2)$ is defined in condition (v) and $m(v, u)$ can be computed explicitly.

Auxiliary objects

dual basis

$$\langle \widetilde{g}_i, g_j \rangle = 2\pi\delta_{ij}.$$

dual cell

$$\widetilde{\mathcal{C}} = \{\theta \in \mathbb{R}^d : \theta = \sum_{j=1}^{d} \theta_j \widetilde{g}_j,\ -1/2 \leqslant \theta_j < 1/2,\ j = 1, \ldots, d\}.$$

$$A(\theta) = \begin{pmatrix} \widetilde{a}_{11}(\theta) + \beta_1 & \widetilde{a}_{12}(\theta) & \cdots & \widetilde{a}_{1p}(\theta) \\ \widetilde{a}_{21}(\theta) & \widetilde{a}_{22}(\theta) + \beta_2 & \cdots & \widetilde{a}_{2p}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_{p1}(\theta) & \widetilde{a}_{p2}(\theta) & \cdots & \widetilde{a}_{pp}(\theta) + \beta_p \end{pmatrix},$$

where $\widetilde{a}_{jk}(\theta)$ is defined by

$$\widetilde{a}_{jk}(\theta) = \sum_{g \in \Gamma} a(v_j + g, v_k) e^{-i\langle g, \theta \rangle} \overset{(vii)}{=} \sum_{g \in \Gamma} a(v_j + g, v_k) \cos\langle g, \theta \rangle.$$

Direct integral decomposition

Let's define an operator $U : \ell^2(\mathbb{Z}^d) \to L^2(\widetilde{\mathcal{C}}, \mathbb{C}^p)$ by

$$(Uf)(v,\theta) = |\widetilde{\mathcal{C}}|^{-1/2} \sum_{g \in \Gamma} e^{-i\langle g,\theta\rangle} f(v+g), \quad v \in \Omega.$$

The operator $\mathcal{A}$ is unitary equivalent to the direct integral of matrices $A(\theta)$

$$U\mathcal{A}U^{-1} = \int_{\widetilde{\mathcal{C}}} \oplus A(\theta)\, d\theta,$$

It means that for every $f \in \ell^2(\mathbb{Z}^d)$

$$U\mathcal{A}f(v,\theta) = A(\theta)Uf(v,\theta).$$

Spectral properties of $\mathcal{A}$

Let the eigenvalues of the matrix family $A(\theta)$ be ordered in non-increasing order for every parameter θ: $\lambda_1(\theta) \geqslant \ldots \geqslant \lambda_p(\theta)$.

$$\sigma(\mathcal{A}) = \bigcup_{j=1}^{p} \bigcup_{\theta \in \widetilde{\mathcal{C}}} \lambda_j(\theta),$$

1. For all $\theta \in \widetilde{\mathcal{C}}$ inequality

$$\lambda_1(0) - \lambda_1(\theta) \geqslant 0$$

   holds, moreover the equality is achieved only for $\theta = 0$.
2. The distance from $\lambda_1(0)$ to the second band of $\mathcal{A}$ is strictly positive

$$\lambda_1(0) - \sup_{\theta \in \widetilde{\mathcal{C}}} \lambda_2(\theta) > 0.$$

Since $M(v,u,t)$ is a solution of (1)

$$M(v,u,t) = e^{\mathcal{A}t}\delta_u(v) = \langle e^{\mathcal{A}t}\delta_u(\cdot), \delta_v(\cdot)\rangle_{\ell^2(\mathbb{Z}^d)}.$$

Since U is unitary

$$M(v,u,t) = \frac{1}{|\widetilde{\mathcal{C}}|}\int_{\widetilde{\mathcal{C}}}\sum_{j=1}^{p} e^{\lambda_j(\theta)t}\, e^{i\langle \gamma_v-\gamma_u,\theta\rangle}\,\overline{\psi}_j(\omega_u,\theta)\,\psi_j(\omega_v,\theta)\,d\theta.$$

Key arguments to asymptotic 1

Theorem (Rytova A. and Yarovaya E. 2016)

Let

$$L(t) = \int\limits_{[-\pi,\pi]^d} f(x) e^{-tS(x)}\, dx,$$

where $f(\cdot)$, $S(\cdot)$ are continuous functions such that $f(0) \neq 0$, $S(x) > 0$ for all $x \neq 0$. Let the following functions be equivalent for $\|x\| \to 0$:

$$S(x) \sim \eta\left(\frac{x}{\|x\|}\right)\|x\|^\alpha,$$

for some $\alpha > 0$ and for positive and continuous function on the unit sphere $\eta(\cdot)$ Then there is such $C > 0$ that the following functions are equivalent for $t \to \infty$

$$L(t) \sim Cf(0)t^{-d/\alpha}.$$

Key arguments to asymptotic 2

Theorem (Kozyakin V. 2016)

Let $\alpha \in (0, 2)$ and

$$F(\theta) = \sum_{z \in \mathbb{Z}^d \setminus \{0\}} a_z (1 - \cos\langle z, \theta \rangle), \quad \theta \in [-\pi, \pi]^d,$$

with a_z satisfying

$$a_z \|z\|_\infty^{d+\alpha} \to 1$$

for $\|z\| \to +\infty$. Then the following functions are equivalent for $\|\theta\| \to 0$

$$F(\theta) \sim \frac{2}{\alpha} \Gamma(1 - \alpha) \cos\left(\frac{\pi\alpha}{2}\right) \|\theta\|^\alpha f\left(\frac{\theta}{\|\theta\|}\right),$$

where $f(\cdot)$ is positive continuous function on an unit sphere.

Thank you for your attention!