On supercritical branching random walks on periodic lattices

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New Trends of Stochastic Analysis – 2021
1–5 June 2021

A random walk

Suppose there is a particle at a point $v \in \mathbb{Z}^d$ at time $t = 0$. This particle will either move to a point $u \neq v$, $u \in \mathbb{Z}^d$ or will remain at v over a short period of time δt.

a probability of the transition $v \to u$

$$p(v, u, \delta t) = a(v, u)\delta t + o(\delta t),$$

a probability of the transition $v \to v$

$$p(v, v, \delta t) = 1 + a(v, v)\delta t + o(\delta t).$$

The transition intensity 1

The value $a(v, u)$ is called the transition intensity between v and u.

(i) $a(v, u) \geqslant 0, \quad v \neq u;$

(ii) $a(v, v) < 0;$

(iii) $\sum\limits_{u \in \mathbb{Z}^d} a(v, u) = 0;$

The transition intensity 2

Let $g_1, \ldots, g_d$ be a family of linearly independent (not necessarily orthogonal) vectors with integer coordinates. By a lattice we mean a set

$$\Gamma = \Big\{ g \in \mathbb{Z}^d : g = \sum_{j=1}^{d} n_j g_j,\ n_j \in \mathbb{Z},\, j = 1, \ldots, d \Big\}.$$

(iv) $a(v,u) = a(u,v) = a(v+g, u+g), \quad \forall g \in \Gamma;$

(v) $\sum_{u \in \mathbb{Z}^d} \|u\|^2 |a(v,u)| < \infty,\ v \in \mathbb{Z}^d;$

(vi) the graph $G = (\mathbb{Z}^d, \mathcal{E})$ with the vertex set $\mathbb{Z}^d$ and edge set

$$\mathcal{E} = \{(v,u) :\ a(v,u) > 0,\ v,u \in \mathbb{Z}^d\}$$

is connected.

A branching source

Suppose there is a particle at a point with branching source and it can't move anywhere from there. We assume that a particle can generate several descendants over a short period of time δt.

a probability of generating $k \neq 1$ descendants

$$p_k = b_k \delta t + o(\delta t).$$

a probability of generating $k = 1$ descendant

$$p_1 = 1 + b_1 \delta t + o(\delta t).$$

The branching intensity

The value b_k is called a branching intensity into k descendants.

(1) $b_k \geqslant 0, \quad k \neq 1;$

(2) $b_1 \leqslant 0;$

(3) $\sum\limits_{k=0}^{+\infty} b_k = 0;$

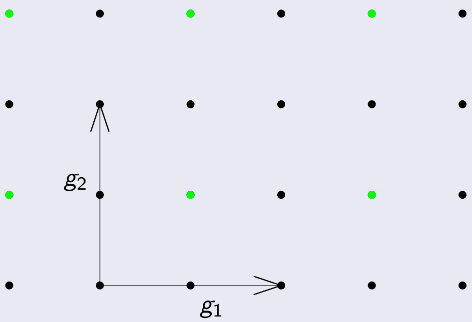
(4) $\beta = \sum\limits_{k=1}^{+\infty} k b_k < \infty;$

Locations of the branching sources

Let branching intensities depend on the point v and value $\beta(v)$ be periodic with respect to the lattice Γ.

(5) $\beta(v+g)=\beta(v), \quad g\in\Gamma.$

Example



$g_1=(2,0)$

$g_2=(0,2)$

A random walk with a periodic set of branching sources

We assume that each new particle evolves according to the same law independently of other particles.
Each particle located at a point $v \in \mathbb{Z}^d$ at time t can either move to a point $u \neq v$ or remain at the source and produce $k \neq 1$ descendants located at the point v (for $k = 0$ we assume that the number of descendants is 0; that is, the particle dies) or remain unchanged (that is, no changes occur) over a short period of time $[t; t + \delta t)$.

a probability of transition $v \to u$

$$p(v, u, \delta t) = a(v, u)\delta t + o(\delta t),$$

a probability of generating $k \neq 1$ offsprings in v

$$p_k(v, \delta t) = b_k(v)\delta t + o(\delta t).$$

a probability of remaining unchanged

$$p(v, \delta t) = 1 + a(v, v)\delta t + b_1(v)\delta t + o(\delta t).$$

Finite number of branching sources

Yarovaya E., et. all [1998-2007]

One source. Asymptotic behaviour of all moments. Limit theorems.

Vatutin V., Topchii V. [2005]

Limit theorem for catalytic BRW on $\mathbb{Z}$ with one source of branching in super critical case.

Rytova A., Yarovaya E. [2018]

Extinction probability for sub and super critical cases.

Khristolyubov I., Yarovaya E. [2019]

N sources. Asymptotic behaviour of all moments in subcritical and supercritical cases. Limit theorems.

Mean value of particles

By $M(v,u,t)$ we denote the mean number of particles at a point u at time t, provided that at the initial time $t=0$ there was one particle at a point v. The function $M(v,u,t)$ satisfies the following Cauchy problem:

The Cauchy problem

$$\begin{cases} M'_t(v,u,t) = \mathcal{A}M(v,u,t), \\ M(v,u,0) = \delta_u(v). \end{cases}$$

The operator $\mathcal{A}$

$$\mathcal{A} = \mathcal{A}_0 + Q,$$

$$(\mathcal{A}_0 f)(v) = \sum_{w \in \mathbb{Z}^d} a(v,w) f(w),$$

$$(Qf)(v) = \beta(v) f(v).$$

Properties of $\mathcal{A}$

The operator $\mathcal{A} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ satisfies the following properties:

- $\mathcal{A}$ is bounded;
- $\mathcal{A}$ is self-adjoint;
- $\mathcal{A}_0$ is non-positive;
- $\mathcal{A}$ is periodic with respect to the lattice Γ.

Connection with a discrete Laplacian

If condition (v) is replaced with stronger condition that for any $v \in \mathbb{R}^d$ there are only finite number of transition probabilities $a(v, u)$ are not zero, then operator $-\mathcal{A}_0$ is a discrete combinatorial Laplacian on the graph G defined in (vi). In this case operator $-\mathcal{A}$ is a discrete Schödinger with periodic potential.

Asymptotic behaviour of $M(v, u, t)$

Theorem

The function $M(v, u, t)$ has the following asymptotic behaviour as $t \to \infty$

$$M(v, u, t) = e^{\lambda_1(0)t} t^{-\frac{d}{2}} c_0(v, u)\Big(1 + O(t^{-1})\Big)$$

where

$$c_0(v, u) = \frac{(2\pi)^{\frac{d}{2}}}{|\widetilde{\mathcal{C}}|} \frac{\psi_1(v', 0)\psi_1(u', 0)}{\sqrt{\left|\det\left\{\frac{\partial^2 \lambda_1(\theta)}{\partial \theta^2}\Big|_{\theta=0}\right\}\right|}}$$

where $v = v' + \gamma_v$, $u = u' + \gamma_u$, $v', u' \in \Omega$, $\gamma_v, \gamma_u \in \Gamma$.

If $a(v, u)$ decays fast enough then

$$M(v, u, t) \stackrel{as}{=} e^{\lambda_1(0)t} t^{-\frac{d}{2}} \sum_{k=0}^{\infty} c_k(v, u) t^{-k}.$$

Notations

Fundamental vertex set

We can choose a set of vertices $\Omega = \{v_1, \ldots, v_p\}$ such that for any $u \in \mathbb{Z}^d$ there is an unique representation

$$u = \omega_u + \gamma_u, \quad \omega_u \in \Omega, \gamma_u \in \Gamma.$$

Dual basis

$$\langle \widetilde{g}_i, g_j \rangle = 2\pi\delta_{ij}.$$

Dual cell

$$\widetilde{\mathcal{C}} = \{\theta \in \mathbb{R}^d : \theta = \sum_{j=1}^{d} \theta_j \widetilde{g}_j, \ -1/2 \leqslant \theta_j < 1/2, \, j = 1, \ldots, d\}.$$

An auxiliary matrix family

$$A(\theta) = \begin{pmatrix} \widetilde{a}_{11}(\theta) + \beta_1 & \widetilde{a}_{12}(\theta) & \cdots & \widetilde{a}_{1p}(\theta) \\ \widetilde{a}_{21}(\theta) & \widetilde{a}_{22}(\theta) + \beta_2 & \cdots & \widetilde{a}_{2p}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_{p1}(\theta) & \widetilde{a}_{p2}(\theta) & \cdots & \widetilde{a}_{pp}(\theta) + \beta_p \end{pmatrix},$$

where the functions $\widetilde{a}_{jk}(\theta)$ and constants β_j is defined by

$$\widetilde{a}_{jk}(\theta) = \sum_{g\in\Gamma} e^{-i\langle g,\theta\rangle} a(v_j + g, v_k), \qquad \beta_j = \beta(v_j).$$

Connection between $\mathcal{A}$ and $A(\theta)$

Let the eigenvalues of the matrix family $A(\theta)$ be ordered in non-increasing order for every parameter θ: $\lambda_1(\theta) \geqslant \ldots \geqslant \lambda_p(\theta)$.

$$\sigma(\mathcal{A}) = \bigcup_{j=1}^{p} \bigcup_{\theta\in\widetilde{\mathcal{C}}} \lambda_j(\theta),$$

Properties of $\lambda_1(\theta)$

Theorem

For $\lambda_1(\theta)$ the following statements hold:

a) *For all $\theta \in \widetilde{\mathcal{C}}$*

$$\lambda_1(0) - \lambda_1(\theta) \geqslant 0.$$

The equality is achieved only for $\theta = 0$.

b) *the distance between the right edge of the spectrum of $\mathcal{A}$ and the right edge of the second spectral band is positive, i.e.*

$$\lambda_1(0) - \sup_{\theta \in \widetilde{\mathcal{C}}} \lambda_2(\theta) > 0;$$

c) *the determinant of the Hessian matrix of $\lambda_1(\theta)$ does not vanish at $\theta = 0$, i.e.*

$$\det \left\{ \frac{\partial^2 \lambda_1(\theta)}{\partial \theta^2} \Big|_{\theta=0} \right\} \neq 0;$$

d) *$\lambda_1(0)$ is not an eigenvalue of $\mathcal{A}$.*

Properties of $\lambda_1(0)$

Theorem

$\lambda_1(0)$ is positive if either $\sum_{j=1}^{p} \beta_j > 0$ or for some $m = 1, \ldots, p$ $\beta_m \geqslant \|A_0(0)\|$.

Theorem

If we add the coupling constant μ then

- *for $\mu \to 0$*

$$\lambda_1(0) = \frac{\mu}{p} \sum_{j=1}^{p} \beta_j + O(\mu^2).$$

- *for $\mu \to \infty$*

$$\lambda_1(0) = \mu \max_{j=1,\ldots,p} \beta_j + O(1).$$

The moment of order n (supercritical case)

Let $M_n(v,u,t)$ be moment of order n for our BRW. Assume that additionally to (4) and (5) the following conditions are satisfied:

$(4')$ $\beta^{(n)}(v) = \sum\limits_{k=n}^{+\infty} k(k-1)\dots(k-n+1)b_k(v) < \infty$;

$(5')$ $\beta^{(k)}(v+g) = \beta^{(k)}(v), \quad g \in \Gamma, \quad l = k,\dots,n.$

The Cauchy problem for M_n

$$\begin{cases} \partial_t M_n(v,u,t) & = (\mathcal{A}M_n)(v,u,t) + R_n(v,u,t), \\ M_n(v,u,0) & = \delta_u(v), \end{cases}$$

where

$$\mathcal{A}f(v) = \sum_{u\in\mathbb{Z}^d} a(v,u)f(u) + \beta(v)f(v),$$

$$R_n(v,u,t) = \begin{cases} \sum\limits_{r=2}^{n} \frac{\beta^{(r)}(v)}{r!} \sum\limits_{\substack{i_1,\dots,i_r>0 \\ i_1+\dots+i_r=n}} \frac{n!}{i_1!\dots i_r!} M_{i_1}(v,u,t)\dots M_{i_r}(v,u,t), & n \geq 2, \\ 0, & n = 1. \end{cases}$$

Asymptotic behaviour of $M_n(v, u, t)$

Theorem

Suppose that $\lambda_1(0) > 0$. Then the function $M_n(v, u, t)$ has the following asymptotic behaviour as $t \to \infty$

$$\ln M_n(v, u, t) = n\lambda_1(0)t - \frac{dn}{2} \ln t + O(1).$$

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Thank you for your attention!