Investigation of a Stochastic Model of Nonlinear Filtration

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A lot of physical, mechanical, and technical processes can be described on the basis equations and systems of equations that are not resolved with respect to the highest derivative.
A lot of boundary value problems for the equations and the systems of equations not resolved with respect to time derivative are considered in the framework of abstract Sobolev type equations

$$L\,\dot{x} + Mx + \sum_{j=1}^{k} N_j(x) = y, \quad \ker L \neq \{0\} \tag{1}$$

that make up the vast field of non-classical equations of mathematical physics.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary of class C^∞. In the cylinder $\Omega \times \mathbf{R}_+$ consider Dirichlet problem

$$x(s,t) = 0, \ (s,t) \in \partial\Omega \times \mathbf{R}_+, \tag{2}$$

for the generalized filtration Boussinesq equation[1]

$$(\lambda - \Delta)x_t = \alpha\Delta(|x|^{p-2}x) + y, \ p \geq 2, \tag{3}$$

with the Cauchy initial condition

$$x(s,0) = x_0(s), \quad s \in \Omega. \tag{4}$$

Equation (3) is the most interesting particular case of the equation obtained by E.S. Dzektser. Here the desired function $x = x(s,t)$ corresponds to the potential of speed of movement of the free surface of the filtered liquid; the parameters $\alpha \in \mathbf{R}_+$, $\lambda \in \mathbf{R}$ characterize the medium, and the parameter λ can take negative values.

[1] Дзекцер, Е.С. Обобщение уравнения движения грунтовых вод со свободной поверхностью / Е.С. Дзекцер // Доклад Академии наук СССР. – 1972. – Т. 202, № 5. – С. 1031–1033.

Structure

- Thematic justification
- Phase space morphology
- Investigation of a Model of Nonlinear Filtration
- Investigation of a Stochastic Model of Nonlinear Filtration

Systematic study of initial-boundary value problems for the equations not resolved with respect to time derivative began in the '40s of the last century with the works of S.L. Sobolev. Currently, they constitute an independent part of theory of nonclassical equations of mathematical physics.

S.L. Sobolev

A.I. Kozhanov, G.V. Demidenko, A.G. Sveshnikov, M.O. Korpusov, and others

N.A. Sidorov, B.V. Loginov, A.V. Sinitsyn, M.V. Falaleev, and others
I.V. Mel'nikova, A.I. Filinkov, M.A. Al'shanskiy, and others

Yu.E. Boyarintsev, V.F. Chistyakov, A.V. Keller, and others

R. Showalter, A. Favini, A. Yagi, and others

G.A. Sviridyuk, T.G. Sukacheva, A.A. Zamyshlyaeva, S.A. Zagrebina, and others

Sviridyuk G.A., Manakova N.A. Dynamic Models of Sobolev Type with the Showalter–Sidorov Condition and Additive "Noises". Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software, **7** (1), 90–103 (2014). (in Russian)

Gliklikh Yu.E., Mashkov E.Yu. Stochastic Leontieff Type Equations in Terms of Current Velocities of the Solution II. Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software, **9** (3), 31–40 (2016).

Favini, A., Sviridyuk, G., Sagadeeva, M.: Linear Sobolev Type Equations with Relatively p-Radial Operators in Space of "Noises". Mediterranean Journal of Mathematics, **12** (6), 4607–4621 (2016). DOI: 10.1007/s00009-016-0765-x

Favini, A., Sviridyuk, G.A., Zamyshlyaeva, A.A.: One Class of Sobolev Type Equations of Higher Order with Additive "White Noise". Communications on Pure and Applied Analysis, **15** (1), 185–196 (2016).

Favini, A., Zagrebina, S.A., Sviridyuk, G.A.: Multipoint Initial-Final Value Problems for Dynamical Sobolev-Type Equations in the Space of Noises. Electronic Journal of Differential Equations, **2018** (128), 1–10 (2018).

Let $\mathbf{H} = W_2^{-1}(\Omega)$, $\mathcal{H} = L_2(\Omega)$, $\mathcal{B} = L_p(\Omega)$ (all functional spaces are defined on domain Ω). Note that there exists the dense and continuous embedding $\overset{\circ}{W}{}_2^1(\Omega) \hookrightarrow L_q(\Omega)$ for $p \geq \frac{2n}{n+2}$, therefore $L_p(\Omega) \hookrightarrow W_2^{-1}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In $\mathbf{H}$, define the scalar product by the formula

$$\langle x, y \rangle = \int_{\Omega} x \tilde{y} ds \quad \forall x, y \in \mathbf{H}, \tag{5}$$

where $\tilde{y}$ is the generalized solution to the homogeneous Dirichlet problem for Laplace operator $(-\Delta)$ in the domain Ω. Let $\mathcal{B}^* = (L_p(\Omega))^*$ and $\mathcal{H}^* = (L_2(\Omega))^*$, where $(L_p(\Omega))^*$ is conjugate space with respect to duality (5). For thus defined $\mathcal{H}^*$ and $\mathcal{B}^*$ there exist dense and continuous embeddings

$$\mathcal{B} \hookrightarrow \mathcal{H} \hookrightarrow \mathbf{H} \hookrightarrow \mathcal{H}^* \hookrightarrow \mathcal{B}^*. \tag{6}$$

Define the operators L and M as follows:

$$\langle Lx, v\rangle = \int\limits_{\Omega} (\lambda x\tilde{v} + xv)ds, \quad x,\, v \in \mathcal{H};$$

$$\langle M(x), v\rangle = -\int\limits_{\Omega} |x|^{p-2} xvds, \quad x,\, v \in \mathcal{B}.$$

Let $\{\varphi_k\}$ be the sequence of eigenfunctions of the homogeneous Dirichlet problem for Laplace operator $(-\Delta)$ in the domain Ω, and $\{\lambda_k\}$ be the corresponding sequence of eigenvalues numbered in non-increasing order taking into account the multiplicity.

Lemma 1. [a] (i) *For all $\lambda \geq -\lambda_1$ the operator $L \in \mathcal{L}(\mathcal{H}; \mathcal{H}^*)$ is self-adjoint, Fredholm, and non-negatively defined, and the orthonormal family $\{\varphi_k\}$ of its functions is total in the space $\mathcal{H}$.* (ii) *Operator $M \in C^1(\mathcal{B}; \mathcal{B}^*)$ is dissipative and p-coercive.*

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[a] Manakova, N.A., Sviridyuk, G.A.: Nonclassical Equations of Mathematical Physics. Phase Space of Semilinear Sobolev Type Equations. Bulletin of the South Ural State University. Series Mathematics. Mechanics. Physics, 8 (3), 31–51 (2016). (in Russian) DOI: 10.14529/mmph160304

If $\lambda \geq -\lambda_1$

$$\ker L = \begin{cases} \{0\}, \text{ if } \lambda > -\lambda_1; \\ \operatorname{span}\{\varphi_1\}, \text{if } \lambda = -\lambda_1. \end{cases}$$

Therefore

$$\operatorname{im} L = \begin{cases} \mathcal{H}^*, \text{ if } \lambda > -\lambda_1; \\ \{x \in \mathcal{H}^* : \langle x, \varphi_1 \rangle = 0\}, \text{ if } \lambda = -\lambda_1, \end{cases}$$

$$\operatorname{coim} L = \begin{cases} \mathcal{H}, \text{ if } \lambda > -\lambda_1; \\ \{x \in \mathcal{H} : \langle x, \varphi_1 \rangle = 0\}, \text{ if } \lambda = -\lambda_1. \end{cases}$$

Hence, the projectors

$$P = Q = \begin{cases} \mathbf{I}, \text{if } \lambda > -\lambda_1; \\ \mathbf{I} - \langle \cdot, \varphi_1 \rangle, \text{if } \lambda = -\lambda_1. \end{cases}$$

Construct the set

$$\mathbf{M} = \begin{cases} \mathcal{B}, \text{ if } \lambda > -\lambda_1; \\ \{x \in \mathcal{B} : \int\limits_\Omega |x|^{p-2} x \varphi_1 \, ds = 0\}, \text{ if } \lambda = -\lambda_1. \end{cases}$$

Theorem 1. Suppose that $p \geq \frac{2n}{n+2}$, $\lambda \geq -\lambda_1$. Then
(i) the set $\mathbf{M}$ is a simple Banach C^1-manifold modelled by the space $\operatorname{coim} L \cap \mathcal{B}$;
(ii) $\forall\, x_0 \in \mathbf{M}$ there exists the unique solution $x \in C^k((0,+\infty);\mathbf{M})$ to problem (2) – (4).

Consider the stationary equation

$$Lx + M(x) + y = 0.$$

Introduce in $\operatorname{coim} L$ norm $|x|^2 = \langle Lx, x \rangle$.
Applying the dissipativity property of operators, we obtain

$$\frac{1}{2}\frac{d}{dt}|x|^2 = \frac{1}{2}\frac{d}{dt}|x-v|^2 \leq \langle M(x) - M(v), x - v \rangle \leq 0,$$

where $v = v(s)$ is the solution of the stationary equation, which in turn is by a stationary solution to equation (1), $x = x(s,t)$ is a solution to equation (1).

Stating of the problem

Consider Sobolev type equation

$$L \overset{o}{\eta} = M(\eta), \tag{7}$$

endowed with the weakened (in the sense of S.G. Krein) Cauchy condition

$$\lim_{t \to 0+} (\eta(t) - \eta_0) = 0. \tag{8}$$

Here $\eta = \eta(t)$ is the desired stochastic process, η_0 is a given random variable, and the symbol $\overset{o}{\eta}$ denotes the Nelson–Gliklikh derivative of the stochastic process $\eta = \eta(t)$

Stating of the problem

Consider a complete probability space $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ and the set of real numbers $\mathbf{R}$ endowed with a Borel σ-algebra.

A random process η is called *continuous*, if almost surely all its trajectories are continuous. Denote by $\mathbf{CL_2}$ the set of continuous random processes, which forms a Banach space.

Definition 1.

(i) Suppose that $\eta \in \mathbf{CL_2}$. The derivative

$$\overset{o}{\eta} = D_S\eta = \frac{1}{2}(D + D_*)\eta = D\eta(t,\cdot) + D_*\eta(t,\cdot) =$$

$$= \lim_{\triangle t \to 0+} E_t^\eta \left(\frac{\eta(t+\triangle t,\cdot) - \eta(t,\cdot)}{\triangle t} \right) + \lim_{\triangle t \to 0+} E_t^\eta \left(\frac{\eta(t,\cdot) - \eta(t-\triangle t,\cdot)}{\triangle t} \right)$$

is called the symmetric mean derivative, where $D\eta(t,\cdot)$ is *derivative on the right* (*on the left* $D_*\eta(t,\cdot)$) *of a random process* η *at the point* $t \in (\varepsilon, \tau)$, if the limit exists in the sense of a uniform metric on $\mathbf{R}$. A random process η is called *mean differentiable on the right (on the left) on* $\mathcal{I}$, if there exists the mean derivative on the right (on the left) at each point $t \in \mathcal{I}$.

Denote the l-th Nelson–Gliklikh derivative of the random process η by $\overset{o}{\eta}{}^{(l)}$, $l \in \mathbf{N}$. Note that the Nelson–Gliklikh derivative coincides with the classical derivative, if $\eta(t)$ is a deterministic function.

Consider the space of "noises" $\mathbf{C}^l\mathbf{L}_2$, $l \in \mathbf{N}$, i.e. the space of random processes from $\mathbf{CL}_2$, whose trajectories are almost surely differentiable by Nelson–Gliklikh on $\mathcal{I}$ up to the order l inclusive.

Consider a real separable Hilbert space $(\mathbf{H}, < \cdot, \cdot >)$ identified with its conjugate space with orthonormal basis $\{\varphi_k\}$. Each element $u \in \mathbf{H}$ can be represented as

$$x = \sum_{k=1}^{\infty} < x, \varphi_k > \varphi_k.$$

Next, choose a monotonely decreasing numerical sequence $K = \{\mu_k\}$ such that

$$\sum_{k=1}^{\infty} \mu_k^2 < +\infty.$$

Consider a sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$, such that $\sum\limits_{k=1}^{\infty} \mu_k^2 \mathbf{D}\xi_k < +\infty$. Denote by $\mathbf{H}_K\mathbf{L}_2$ the Hilbert space of *random K-variables* having the form $\xi = \sum\limits_{k=1}^{\infty} \mu_k \xi_k \varphi_k$. Moreover, a random K-variable $\xi \in \mathbf{H}_K\mathbf{L}_2$ exists, if, for example, $\mathbf{D}\xi_k < \mathrm{const}\ \forall k$. Note that space $\mathbf{H}_K\mathbf{L}_2$ is a Hilbert space with scalar product

$$(\xi^1, \xi^2) = \sum_{k=1}^{\infty} \mu_k^2 \mathbf{E}\xi_k^1 \xi_k^2.$$

Consider a sequence of random processes $\{\eta_k\} \subset \mathbf{CL}_2$ and define $\mathbf{H}$-*valued continuous stochastic* K-*process*

$$\eta(t) = \sum_{k=1}^{\infty} \mu_k \eta_k(t) \varphi_k \tag{9}$$

if series (9) converges uniformly by the norm $\mathbf{H}_K\mathbf{L}_2$ on any compact set in $\mathcal{I}$. Consider the Nelson–Gliklikh derivatives of random K-process

$$\overset{o}{\eta}{}^{(l)}(t) = \sum_{k=1}^{\infty} \mu_k \overset{o}{\eta}{}_k^{(l)}(t) \varphi_k$$

on the assumption that there exist the Nelson–Gliklikh derivatives up to the order l inclusive in the right-hand side, and all series converge uniformly according to the norm $\mathbf{H}_K\mathbf{L}_2$ on any compact from $\mathcal{I}$.

Consider the space $\mathbf{C}(\mathcal{I};\mathbf{H}_K\mathbf{L}_2)$ of continuous stochastic K-processes and the space $\mathbf{C}^l(\mathcal{I};\mathbf{H}_K\mathbf{L}_2)$ of stochastic K-processes whose trajectories are almost surely continuously differentiable by Nelson–Gliklikh up to the order $l \in \mathbf{N}$ inclusive.

Consider dual pairs of reflexive Banach spaces $(\mathcal{H},\mathcal{H}^*)$ and $(\mathcal{B},\mathcal{B}^*)$, such that embeddings (6) are dense and continuous. Let an operator $L \in \mathcal{L}(\mathcal{H};\mathcal{H}^*)$ be linear, continuous, self-adjoint, non-negative defined Fredholm operator, and an operator $M \in C^k(\mathcal{B};\mathcal{B}^*), k \geq 1$, be dissipative. In space $\mathbf{H}$ choose an orthonormal basis $\{\varphi_k\}$ so that

$$\operatorname{span}\{\varphi_1,\ \varphi_2, ..., \varphi_l\} = \ker L, \quad \dim\ker L = l$$

and the following condition holds: $\{\varphi_k\} \subset \mathcal{B}$. Taking into account that the operator L is self-adjoint and Fredholm, we identify $\mathbf{H} \supset \ker L \equiv \operatorname{coker} L \subset \mathbf{H}^*$ and, similarly, construct the space $\mathbf{H}^*_K\mathbf{L}_2$ according to the corresponding orthonormal basis. We use the subspace $\ker L$ in order to construct the subspace $[\ker L]_K\mathbf{L}_2 \subset \mathbf{H}_K\mathbf{L}_2$ and, similarly, the subspace $[\mathrm{coker\,L}]_{\mathrm{K}}\mathbf{L}_2 \subset \mathbf{H}^*_{\mathrm{K}}\mathbf{L}_2$.

Taking into account that embeddings (6) are continuous and dense, we construct the spaces

$$\mathcal{H}^*_K\mathbf{L}_2 = [\operatorname{coker} L]_K\mathbf{L}_2 \oplus [\operatorname{im} L]_K\mathbf{L}_2 \text{ and } \mathcal{B}^*_K\mathbf{L}_2 = [\operatorname{coker} L]_K\mathbf{L}_2 \oplus [\overline{\operatorname{im} L}]_K\mathbf{L}_2.$$

We use the subspace $\mathrm{coim}\, L \subset \mathcal{H}$ in order to construct the subspace $[\mathrm{coim}\, L]_K \mathbf{L}_2$ such that the space

$$\mathcal{H}_K \mathbf{L}_2 = [\ker L]_K \mathbf{L}_2 \oplus [\mathrm{coim}\, L]_K \mathbf{L}_2.$$

Denote $[\ker L]_K \mathbf{L}_2 \equiv \mathcal{B}_K^0 \mathbf{L}_2$ such that the space $\mathrm{coim}\, L \cap \mathcal{B}$ in order to construct the set $\mathcal{B}_K^1 \mathbf{L}_2$, then

$$\mathcal{B}_K \mathbf{L}_2 = \mathcal{B}_K^0 \mathbf{L}_2 \oplus \mathcal{B}_K^1 \mathbf{L}_2.$$

The following Lemma is correct, since the operator L is self-adjoint and Fredholm.

Lemma 2.

(i) Let operator $L \in \mathcal{L}(\mathcal{H}; \mathcal{H}^*)$ be a linear, continuous, self-adjoint, non-negatively defined Fredholm operator, then the operator $L \in \mathcal{L}(\mathcal{H}_K \mathbf{L}_2; \mathcal{H}_K^* \mathbf{L}_2)$, and

$$\mathbf{H}_K \mathbf{L}_2 \supset [\ker L]_K \mathbf{L}_2 \equiv [\mathrm{coker}\, L]_K \mathbf{L}_2 \subset \mathbf{H}_K^* \mathbf{L}_2$$

if

$$\mathbf{H} \supset \ker L \equiv \mathrm{coker}\, L \subset \mathbf{H}^*.$$

(ii) There exists a projector Q of the space $\mathcal{B}_K^* \mathbf{L}_2$ on $[\overline{\mathrm{im}\, L}]_K \mathbf{L}_2$ along $[\mathrm{coker}\, L]_K \mathbf{L}_2$.
(iii) There exists a projector P of the space $\mathcal{B}_K \mathbf{L}_2$ on $\mathcal{B}_K^1 \mathbf{L}_2$ along $\mathcal{B}_K^0 \mathbf{L}_2$.

Suppose that $\mathcal{I} \equiv (0, +\infty)$. We use the space $\mathbf{H}$ in order to construct the spaces of K-"noises" spaces $\mathbf{C}^k(\mathcal{I}; \mathbf{H}_K\mathbf{L}_2)$ and $\mathbf{C}^k(\mathcal{I}; \mathcal{B}_K\mathbf{L}_2),\ k \in \mathbf{N}$. Consider the stochastic Sobolev type equation

$$L \overset{o}{\eta} = M(\eta). \tag{10}$$

A solution to equation (10) is a stochastic K-process. Stochastic K-processes $\eta = \eta(t)$ and $\zeta = \zeta(t)$ are considered to be equal, if almost surely each trajectory of one of the processes coincides with a trajectory of other process.

Definition 2.

A stochastic K-process $\eta \in C^1(\mathcal{I}; \mathcal{B}_K\mathbf{L}_2)$ is called a *solution to equation* (10), if almost surely all trajectories of η satisfy equation (10) for all $t \in \mathcal{I}$. A solution $\eta = \eta(t)$ to equation (10) that satisfies the initial value condition

$$\lim_{t \to 0+} (\eta(t) - \eta_0) = 0 \tag{11}$$

is called a *solution to Cauchy problem* (10), (11), if the solution satisfies condition (11) for some random K-variable $\eta_0 \in \mathcal{B}_K\mathbf{L}_2$.

Fix $\omega \in \Omega$. Let $\eta = \eta(t), t \in \mathcal{I}$ be a solution to equation (10), then η belongs to the set

$$\mathbf{M} = \begin{cases} \{\eta \in \mathcal{B}_K \mathbf{L}_2 : (\mathbf{I} - Q)M(\eta) = 0\}, & \text{if } \ker L \neq \{0\}; \\ \mathcal{B}_K \mathbf{L}_2, & \text{if } \ker L = \{0\}. \end{cases} \tag{12}$$

Next, we consider the Dirichlet problem

$$\eta(s,t) = 0, \ (s,t) \in \partial\Omega \times \mathbf{R}_+, \tag{13}$$

for the stochastic Boussinesq equation

$$(\lambda - \Delta) \overset{o}{\eta} = \Delta(|\eta|^{p-2}\eta), \ p \geq 2. \tag{14}$$

Theorem 2.

Let $p \geq \frac{2n}{n+2}$, $\lambda \geq -\lambda_1$. Then for any $\eta_0 \in \mathbf{M}$ exists a solution $\eta \in \mathbf{C^1 L_2}(\mathcal{I}; \mathbf{M})$ of problem (11), (13), (14).

Thank you for your attention!