On a coupled Price-Liquidity equity model with regime switching

Nadezhda P. Krasii [1]

[1]DSTU, **Rostov-on-Don**, Russian Federation

Joint work with **Manuel L. Esquível** and **Pedro P. Mota**

International Conference **New Trends of Stochastic Analysis - 2021**, *Divnomorskoe, June 1 to 5*

Outline

Introduction – i – Motivation

Example of co-variations between price and volume of traded shares of the asset

Introduction – ii – Context

Goal, method, and an approach to *quantitative liquidity*

Coupling **price-liquidity** by the regime switching

Price and liquidity variable regime switching occurs at the stopping times corresponding to a change of region in the product state space of price and liquidity.

Quantitative proxies for liquidity

Bid-Ask spread or number of shares traded by unit of time.

Main goal of this work

Present a rigorous development of a coupled model for joint evolution **price-liquidity** using stochastic differential equations and regime switching.

The Model – i – Stochastic Differential Equations

Equations for price S_t and liquidity L_t, thresholds, coupling by regime switching

Two SDE - for price S_t and for liquidity L_t

$$\begin{cases} dS_t = \mu(t, S_t, \boldsymbol{\theta})dt + \sigma(t, S_t, \boldsymbol{\theta})dW_t, \ S_0 \in \mathbb{R}^+ \\ dL_t = \nu(t, L_t, \boldsymbol{\lambda})dt + \eta(t, L_t, \boldsymbol{\lambda})dW_t, \ L_0 \in \mathbb{R}^+ \end{cases}$$

$(W_t)_{t\geq 0}$ Brownian process, $\boldsymbol{\theta} \in \{\boldsymbol{\theta}^h, \boldsymbol{\theta}^s, \boldsymbol{\theta}^d\}$, $\boldsymbol{\lambda} \in \{\boldsymbol{\lambda}^h, \boldsymbol{\lambda}^s, \boldsymbol{\lambda}^d\}$,

Price **drift** regime switching – liquidity – thresholds L_m and L_M

$$\mu(t, S_t, \boldsymbol{\theta}) = \begin{cases} \mu(t, S_t, \boldsymbol{\theta}^h) & \text{if } L_t > L_M \\ \mu(t, S_t, \boldsymbol{\theta}^s) & \text{if } L_m \leq L_t \leq L_M \\ \mu(t, S_t, \boldsymbol{\theta}^d) & \text{if } L_t < L_m \ , \end{cases}$$

similar relation for price **volatility** $\sigma(t, S_t, \boldsymbol{\theta})$

The Model – ii – Coupling and possible scenarios

Coupling main idea: price threshold hitting implies liquidity regime switching

Liquidity **drift** regime switching – price – thresholds S_m and S_M

$$\nu(t, L_t, \boldsymbol{\lambda}) = \begin{cases} \nu(t, L_t, \boldsymbol{\lambda}^h) & \text{if } S_t > S_M \\ \nu(t, L_t, \boldsymbol{\lambda}^s) & \text{if } S_m \leq S_t \leq S_M \\ \nu(t, L_t, \boldsymbol{\lambda}^d) & \text{if } S_t < S_m \ , \end{cases}$$

similar relation for liquidity **volatility** $\eta(t, S_t, \boldsymbol{\theta})$

Price and Liquidity co-influences scenarios

Scenarios	I	II	III	IV	V	VI	VII	VIII
Liquidity on highest price subdomain	↑	↓	↓	↓	↓	↓	↓	↓
Liquidity on lowest price subdomain	↓	↓	↓	↓	↓	↑	↑	↑
Price on highest liquidity subdomain	↓	↓	↓	↑	↑	↓	↓	↑
Price on lowest liquidity subdomain	↑	↓	↑	↓	↑	↓	↑	↓

The Model – iii – Analysis of first scenario

Mutual behaviour and respective drift properties translation

Influence of price variations on liquidity variations and vice-versa

1. Price larger than highest threshold $\Rightarrow$ liquidity goes up.
2. Price smaller than lowest threshold $\Rightarrow$ liquidity goes down.
3. Liquidity larger than highest threshold $\Rightarrow$ price goes down.
4. Liquidity smaller than lowest threshold $\Rightarrow$ price goes up.

Implementation of first scenario on price and liquidity drifts

- $\mu(t, S_t, \boldsymbol{\theta}^h) < 0$, $\mu(t, S_t, \boldsymbol{\theta}^d) > 0$
- $\mu(t, S_t, \boldsymbol{\theta}^s) \ll \min\left(\left|\mu(t, S_t, \boldsymbol{\theta}^h)\right|, \left|\mu(t, S_t, \boldsymbol{\theta}^d)\right|\right)$
- $\nu(t, L_t, \boldsymbol{\lambda}^h) > 0$, $\nu(t, L_t, \boldsymbol{\lambda}^d) < 0$
- $\nu(t, L_t, \boldsymbol{\lambda}^s) \ll \min\left(\left|\nu(t, L_t, \boldsymbol{\lambda}^h)\right|, \left|\nu(t, L_t, \boldsymbol{\lambda}^d)\right|\right)$

The Model – iv – A first scenario price-liquidity trajectory

Trajectory evolution according with price and liquidity drifts signs

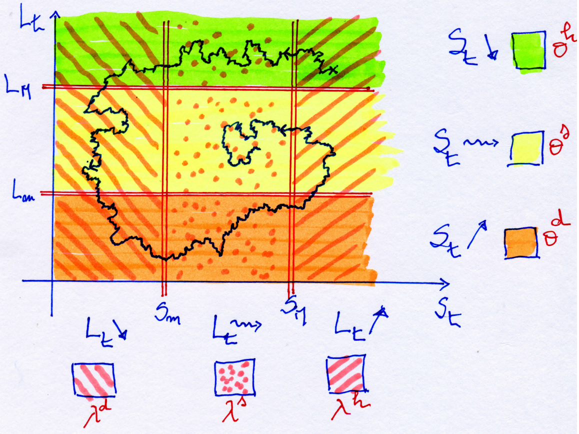

Figure: Possible evolution of $(S_t, L_t)_{t\geq 0}$ on first scenario

The Model – v – A double threshold crossing

Main idea for the regime switching times: hitting times of inner boundary

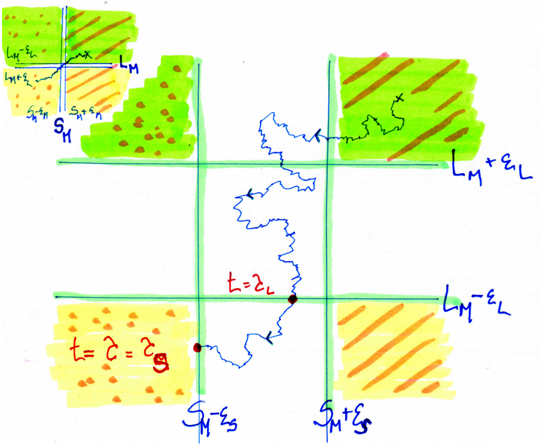

Figure: $(S_t)_{t \geq 0}$ and $(L_t)_{t \geq 0}$ hitting time of inner boundary

Existence and unicity – i – The regime switching times

Defining the regime switching times and statement of an essential property

Regime switching times as hitting times of a inner boundary

- Thresholds S_m, S_M, L_m and L_M divide $[0, +\infty[\times[0, +\infty[$ in 9 subdomains.
- With $\{S_M - \epsilon_S, S_M + \epsilon_S\}$, $\{L_M - \epsilon_L, L_M + \epsilon_L\}$ etc, double thresholds, regime switching stopping times are **hitting times** on the **inner boundary** of the subdomain.

Crucial property: non accumulation points in compact intervals

$0 \equiv \tau_0 < \tau_1 < \cdots < \tau_n < \cdots$ is increasing, $\lim_{n\to+\infty} \tau_n = +\infty$ a.s.

$$\forall T \in \mathbb{R}_+ \,, \quad \# \{k \geq 1 : \tau_k(\omega) \leq T\} < +\infty \text{ a.s. } .$$

i.e., a.s. $(\tau_n(\omega))_{n\geq 0}$ don't have accumulation points in $[0, T]$.

Existence and unicity – ii – strong solutions, unicity in law

A theorem for SDE with possible irregular coefficients

Yamada-Watanabe Theorem (1971), SDE with $\alpha = \mu, \nu$; $\beta = \sigma, \eta$

α and β **progressively measurable**; $\rho_1 : [0, +\infty[\mapsto [0, +\infty[$, **increasing** and continuous such that $\rho_1(0) = 0$ and:

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{+\infty} \frac{du}{\rho_1(u)} = +\infty \ , \ (\beta(t, x, \boldsymbol{\theta}) - \beta(t, y, \boldsymbol{\theta}))^2 \leq \rho_1(|x - y|) \ ;$$

$\rho_2 : [0, +\infty[\mapsto [0, +\infty[$, **increasing and concave** function, such that $\rho_2(0) = 0$, and:

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{+\infty} \frac{du}{\rho_2(u)} = +\infty \ , \ (\alpha(t, x, \boldsymbol{\theta}) - \alpha(t, y, \boldsymbol{\theta})) \leq \rho_2(|x - y|) \ .$$

Then, there is **strong existence** and **uniqueness in law**.

Existence and unicity – iii – of regime switching processes

Induction using Yamada-Watanabe theorem and non accumulation stopping times

Main idea of a constructive approach to define the solution process

Apply Yamada-Watanabe to solve for $(X^1_t)_{t\in[0,\tau_1]}$, unique in law.

$$\begin{cases} dX^1_t = \alpha(t, X^1_t, \boldsymbol{\theta}_i)dt + \beta(t, X^1_t, \boldsymbol{\theta}_i)dW_t\ , \quad t \in [0, \tau_1]\ , \\ X^1_0 = X_0\ , \end{cases}$$

take regime switching at time τ_1 with new value of the parameter $\boldsymbol{\theta}_j$ and solve for $(X^2_t)_{t\in[\tau_1,\tau_2[}$, unique in law,

$$\begin{cases} dX^2_t = \alpha(t, X^2_t, \boldsymbol{\theta}_j)dt + \beta(t, X^2_t, \boldsymbol{\theta}_j)dW_t\ , \quad t \in [\tau_1, \tau_2]\ , \\ X^2_{\tau_1} = X^1_{\tau_1}\ , \end{cases}$$

so on and so forth. The solution is: $X_t = \sum_{n=1}^{+\infty} X^n_t \mathbb{1}_{[\tau_{n-1},\tau_n[}(t)$.

Main Selected References

- Esquível, M.L. Krasii, N.P. Mota, P.P. **Auto and externally induced regime switching diffusions**. *Communications on Stochastic Analysis* 2020, 14, 27–47.
- Riançо, N.S. Esquível, M.L. Mota, P. Veiga, C. **On a Price-Liquidity Threshold Regime Swicthing Model** (preliminary report). *Proceedings of the 6th St. Petersburg Workshop on Simulation*, Ermakov, S.M.; Melas, V.B.; Pepelyshev, A.N., Eds. VVM com. Ltd 2009 Vol. **I** pp. 419–424.
- Esquível, M.L. Mota, P.P. **On some auto-induced regime switching double-threshold glued diffusions**. *J. Stat. Theory Pract.* 2014, 8, 760–771.
- Yamada, T.; Watanabe, S. **On the uniqueness of solutions of stochastic differential equations**. *J. Math. Kyoto Univ.* 1971, 11, 155–167.

Acknowledgements

Institutions and Funding

Funding Institutions and Instruments

1- Russian Foundation for Basic Research Grant n. 19-01-00451

2- FCT MCT Project UID/MAT/00297/2020 and