Martingales with respect to special filtrations

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Brief description

We introduce a stochastic model of a filtered probability space, that includes a classical example of Dellacherie (1970) from a general theory of stochastic processes. The main result is a description of all local martingales in this model together with a classification whether a local martingale is a martingale, or a uniformly integrable martingale etc. We also give a description of all σ-martingales. This model provides simple examples of

- local martingales which are not martingales;
- martingales which have not a càdlàg modification.

In contrast to Dellacherie's example, this model admits

- local martingales which are not local martingales with respect to the filtration they generate;
- σ-martingales which are not local martingales.

Outline

- Introduction: Some classes of (distributions of) random variables
- Introduction: Max-continuous local submartingales
- Definition of the model and preliminaries
- Characterization and description of martingales, local martingales and σ-martingales. Canonical decomposition of special semimartingales

Introduction: Some classes of (distributions of) random variables

Assume that (L, W, V) is a triple of random variables on some probability space (in fact, we deal essentially with joint distributions of them) satisfying the following relations:

$$W \geqslant 0, \quad V \geqslant 0, \quad L = W - V.$$

Let

$$\Psi_{L,W,V}(t) := \mathsf{E}\big(W \wedge t - V \mathbb{1}_{\{t \geqslant W\}}\big), \quad t \geqslant 0.$$

Equivalently,

$$\Psi_{L,W,V}(t) = t\mathsf{P}(t < W) + \mathsf{E}\big(L \mathbb{1}_{\{t \geqslant W\}}\big).$$

Three conditions

Introduce the following conditions:

$$\Psi_{L,W,V}(t) \geqslant 0 \quad \text{for every } t \geqslant 0, \tag{1}$$

$$\Psi_{L,W,V}(0) = 0 \quad \text{and} \quad \Psi_{L,W,V}(t) \quad \text{is increasing in } t \geqslant 0, \tag{2}$$

$$\Psi_{L,W,V}(t) = 0, \quad \text{for every } t \geqslant 0. \tag{3}$$

It is clear that (3)$\Rightarrow$(2)$\Rightarrow$(1).

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Example

Let $B = (B_t)_{t\in\mathbb{R}_+}$ be a Brownian motion on some stochastic basis, $B_0 = 0$, T a finite stopping time. Put

$$L = B_T, \quad W = \sup_{s\leqslant T} B_s, \quad V = \sup_{s\leqslant T} B_s - B_T.$$

Then (L, W, V) satisfies (2).

If $\mathrm{E}|B_T| < \infty$ and $\mathrm{E}B_T \leqslant 0$ then (L, W, V) satisfies (3) if and only if T is a minimal stopping time, i.e. for any stopping time S with $S \leqslant T$ a.s. and $\mathrm{Law}(B_S) = \mathrm{Law}(B_T)$ we have $S = T$ a.s.

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Converse statement

Rogers (1993), given (L, W, V) satisfying (2), constructed explicitly a stopping time T such that $\mathrm{Law}(L, W) = \mathrm{Law}(B_T, \sup_{s \leqslant T} B_s)$.
We will present another approach to this problem.

Examples

Example

Let X be a nonnegative local submartingale with the compensator A, $A_\infty < \infty$ a.s. Then (1) is satisfied for $W = A_\infty$, $V = X_\infty$.

Proposition (G. (2018))

Assume that a triple (L, W, V) satisfies (1) and $\mathsf{E}L^- \geqslant \mathsf{E}L^+$. Then there is a random variable Z such that $0 \leqslant Z \leqslant W \wedge V$ and the triple $(L, W - Z, V - Z)$ satisfies (3).

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Problem

Given: a distribution of L with $\mathsf{E}L = 0$
To find: a distribution of (L, W, V) which satisfies (3) and maximizes (wrt stochastic order) W.

The solution is given by the Hardy–Littlewood maximal function: define

$$Q_W(u) = \frac{1}{1-u} \int_u^1 Q_L(t)\,dt, \quad u \in (0,1),$$

where $Q_L(t)$ is the quantile function of L. The solution is given by $L := Q_L(U)$ and $W := Q_W(U)$, where U is uniformly distributed on $(0,1)$.

Second and third extreme problem

Problem

Given: a distribution of L
To find: a distribution of (L, W, V) which satisfies (2) and *minimizes* *(wrt stochastic order) W.*

Problem

Given: a distribution of V with $\mathsf{E}V < \infty$
To find: a distribution of (L, W, V) which satisfies (3) and *maximizes* *(wrt convex order) W.*

The solution was firstly found in (G. 2018): Let $Q_V(u)$ be the quantile function of V, then define

$$Q_W(u) = \int_0^u \frac{Q_V(t)}{1-t}\, dt, \quad u \in (0, 1).$$

The solution is given by $V := Q_V(U)$ and $W := Q_W(U)$, where U is uniformly distributed on $(0, 1)$.

Introduction: Max-continuous local submartingales

Notation, max-continuous processes

Given a process $X = (X_t)_{t \geqslant 0}$, the running maximum process is denoted by $\overline{X} = (\overline{X}_t)_{t \geqslant 0}$:

$$\overline{X}_t := \sup_{s \leqslant t} X_s.$$

A process X is max-continuous if the process $\overline{X}$ is continuous.

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Proposition

Let X be a max-continuous local submartingale, $X_0 = 0$, $\mathsf{P}(\overline{X}_\infty < \infty) = 1$. Then X_t converges a.s. to a finite limit X_∞. Put $C_s := \inf\{t : \overline{X}_t > s\}$. The transformation of the process X by the time-change C, i.e., the process $Y := X \circ C := (X_{C_t})_{t \geqslant 0}$ is a max-continuous submartingale relative to the filtration $(\mathscr{F}_{C_t})_{t \geqslant 0}$. Moreover,

$$Y_t = W \wedge t - V \mathbb{1}_{\{t \geqslant W\}},$$

where the r.v.'s W and V are defined by

$$W = \overline{X}_\infty, \quad V = \overline{X}_\infty - X_\infty.$$

In particular, $Y_\infty = X_\infty$ and $\overline{Y}_\infty = \overline{X}_\infty$.

Note that

$$E(Y_t) = \Psi_{L,W,V}(t).$$

In particular, under the assumptions of the proposition, Y is a submartingale and, hence, its expectation is increasing, i.e., condition (2) is satisfied.

Observation

Recall that $Y_t = W \wedge t - V \mathbb{1}_{\{t \geqslant W\}}$. Assume that (2) holds with nonnegative W and V. Then $Y_t - Y_s = (Y_t - Y_s)\mathbb{1}_{\{s<W\}}$, and its conditional expectation given σ-algebra which has the set $\{s < W\}$ as an atom, is equal to $\mathsf{E}(Y_t - Y_s)/\mathsf{P}(s < W)$ on $\{s < W\}$ and 0 outside. This shows that the process Y_t is a submartingale with respect to any filtration $(\mathcal{G}_t)$ with respect to which it is adapted and which has the set $\{t < W\}$ as an atom of $\mathcal{G}_t$. This filtration is the main object of this talk.

It can be proved that if X is a max-continuous local submartingale and Y has the same finite-dimensional distributions as X then Y is a max-continuous local submartingale with respect to the filtration it generates.

Observation

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A question

We may conclude that the set of joint distributions of $(X_\infty, \overline{X}_\infty)$, where X runs over the set of max-continuous local submartingales, coincides with set of joint distributions of r.v.'s (L, W) satisfying (2) (and $W \geqslant L \vee 0$).

A natural question is whether a pair (L, W) satisfying (2) can be realized as the joint distribution of $(X_\infty, \overline{X}_\infty)$, where X is a stopped Brownian motion: $X = B^T$, where $B = (B_t)_{t \in \mathbb{R}_+}$ is a Brownian motion on some stochastic basis, $B_0 = 0$, T a finite stopping time.

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Answer is positive

Rogers (1993) showed that it is always the case. He gives an explicit construction of a required stopping time T.

Alternatively, let

$$Y_t = t\mathbb{1}_{\{t<W\}} + L\mathbb{1}_{\{t\geqslant W\}}$$

be a submartingale. If it is a martingale, then, by the first theorem of Monroe (1972), there is a Brownian motion $B = (B_t)_{t\geqslant 0}$ and a time change $\{\sigma_s\}_{s\geqslant 0}$ such that all σ_s are minimal and

$$\text{Law}(Y_t)_{t\geqslant 0} = \text{Law}(B_{\sigma_t})_{t\geqslant 0}.$$

In general case, the same is true by the generalization of Monroe's theorem due to G. & Urusov (2019). Putting $S = \inf\{s\colon B_{\sigma_s} < s\}$ and $T = \sigma_S$, it can be deduced from the minimality of σ_s that

$$\sup_{s\leqslant S} B_{\sigma_s} = \sup_{t\leqslant T} B_t$$

and hence $\overline{B}_T = S$ and $\text{Law}(B_T, \overline{B}_T) = \text{Law}(L, W)$.

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Definition

A max-continuous local submartingale X, $X_0 = 0$, is called a local max-level martingale if $\mathsf{E}X_{C_t} \equiv 0$, *where* $C_s := \inf\{t : \overline{X}_t > s\}$.

Remark

It can be easily checked that if X is a local max-level martingale, then $\mathsf{E}(X_\infty^-) \geqslant \mathsf{E}(X_\infty^+)$.

In the next theorem we get, in particular, necessary and sufficient conditions for a max-continuous local submartingale X, $X_0 = 0$, to be a closed submartingale or a closed supermartingale (and, respectively, a uniformly integrable martingale). It is interesting to note that these conditions are expressed in terms of the joint law $\mathrm{Law}(X_\infty, \overline{X}_\infty)$ of the terminal value of the process and its maximum.

Theorem

Let X be a max-continuous local submartingale, $X_0 = 0$. Then

(i) *X is a closed submartingale if and only if*

$$\lim_{s\to\infty} s\mathsf{P}(\overline{X}_\infty > s) = 0 \tag{4}$$

and $\mathsf{E}X_\infty^+ < \infty$;

(ii) *X is a local max-level martingale if and only if*

$$\mathsf{E}\big(X_\infty \mathbb{1}_{\{\overline{X}_\infty \leqslant s\}}\big) + s\mathsf{P}\big(\overline{X}_\infty > s\big) = 0 \quad \text{for any } s \geqslant 0; \tag{5}$$

(iii) *X is a closed submartingale if and only if condition* (5) *is satisfied and* $\mathsf{E}(X_\infty^-) < \infty$;.

(iv) *X is a uniformly integrable martingale if and only conditions* (5), (4) *are satisfied and* $\mathsf{E}(X_\infty^-) < \infty$.

Example: Distribution of the maximum of a max-continuous local martingale

The maximum of a max-continuous local submartingale starting from zero may have any distribution on $[0, \infty)$. (It is assumed here and below that the maximum is finite a.s.) However, this is not so for local max-level martingales and, a fortiori, for closed supermartingales and uniformly integrable martingales. For the last two classes of (continuous) processes, the corresponding families of distributions were described by Vallois (1994) in different form.
Given a probability measure ν on $[0, \infty]$, we set $G(x) := \nu([0, x])$, $\overline{G}(x) := 1 - G(x)$, $t_G = \operatorname{ess\,sup} \nu := \sup\{x \geqslant 0 \colon G(x) < 1\}$.

Theorem

Let ν be a probability measure on $[0, \infty)$.

(a) *A necessary and sufficient condition that there exist a local max-level martingale X, whose maximum $\overline{X}_\infty$ has the distribution ν, is that the Lebesgue measure be absolutely continuous with respect to ν on $[0, t_G)$.*

(b) *A necessary and sufficient condition that there exist a local max-level martingale X which is a closed supermartingale and whose maximum $\overline{X}_\infty$ has the distribution ν, is that the Lebesgue measure be absolutely continuous with respect to ν on $[0, t_G)$ and the function $x \rightsquigarrow x\overline{G}(x)$ have finite variation on $[0, \infty)$.*

Theorem (continuation)

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(c) *A necessary and sufficient condition that there exist a uniformly integrable max-continuous martingale* X, $X_0 = 0$, *whose maximum* $\overline{X}_\infty$ *has the distribution* ν, *is that the Lebesgue measure be absolutely continuous with respect to* ν *on* $[0, t_G)$, *the function* $x \rightsquigarrow x\overline{G}(x)$ *have finite variation on* $[0, \infty)$, *and*

$$\lim_{x\to\infty} x\overline{G}(x) = 0.$$

Definition of a single jump filtration and preliminary results

Notation

$$\mathbb{R}_+ = [0, +\infty), \quad \overline{\mathbb{R}}_+ = [0, +\infty].$$

We always assume that there are given a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and a random variable γ with values in $\overline{\mathbb{R}}_+$ on it. Then

$$G(t) = \mathsf{P}(\gamma \leqslant t), t \in \mathbb{R}_+,$$

stands for the distribution function of γ and $\overline{G}(t) = 1 - G(t)$. Put also

$$t_G = \sup\{t \in \mathbb{R}_+ \colon G(t) < 1\}, \quad \mathfrak{T} = \{t \in \mathbb{R}_+ \colon \mathsf{P}(\gamma \geqslant t) > 0\}.$$

We tacitly assume that $\mathsf{P}(\gamma > 0) > 0$.

Assumptions

$\mathsf{P}(0 < \gamma < \infty) = 1$, $t_G = +\infty$.

A filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is defined as the smallest (completed) filtration with respect to which γ is a stopping time (or, equivalently, the process $\mathbb{1}_{\{t\geqslant\gamma\}}$ is adapted), $\mathscr{F} = \mathscr{F}_\infty$, i.e. $\mathscr{F}$ is the completion of $\sigma\{\gamma\}$.

It is shown that the compensator of $\mathbb{1}_{\{t\geqslant\gamma\}}$ has the form

$$A_t = \int_0^{\gamma\wedge t} \frac{dG(s)}{\overline{G}(s-)}.$$

In particular, if G is continuous, then

$$A_t = \log \frac{1}{\overline{G}(\gamma\wedge t)}.$$

Jacod (1975) used this formula for calculating compensators of marked point processes.

Dellacherie (1970)

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It is shown that every local martingale can be represented as a Lebesgue-Stieltjes integral with respect to the local martingale $\mathbb{1}_{\{t\geqslant\gamma\}} - A_t$.

The main idea: if (M_t) is a uniformly integrable martingale, then M_∞ can be represented as $H(\gamma)$ with some function H. Then the structure of the filtration implies that $M_t = H(\gamma)$ for $t \geq \gamma$, and M_t coincides with a deterministic function for $t < \gamma$. Hence, it is enough to prove the representation only at time γ.

Main definition

Our purpose is to define a single jump filtration in such a way that all randomness appears at a random time γ but there are more random events in $\mathscr{F}$ than in $\sigma\{\gamma\}$.

Definition

$\mathscr{F}_t$, $t \in \mathbb{R}_+$, is the collection of subsets A of Ω such that $A \in \mathscr{F}$ and $A \cap \{t < \gamma\}$ is either $\varnothing$ or coincides with $\{t < \gamma\}$.

Lemma

(i) *$\mathscr{F}_t$ is a σ-algebra and a random variable ξ is $\mathscr{F}_t$-measurable, $t \in \mathbb{R}_+$, if and only if ξ is constant on $\{t < \gamma\}$. ξ is $\mathscr{F}_\infty$-measurable if and only if ξ is constant on $\{\gamma = \infty\}$.*

(ii) *The family $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is increasing and right-continuous, i.e. $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is a filtration.*

(iii) *γ is a stopping time and $\mathscr{F}_\gamma = \mathscr{F}_\infty$.*

We call this filtration a single jump filtration generated by γ and $\mathscr{F}$.

Proposition

A random variable T with values in $\overline{\mathbb{R}}_+$ is a stopping time if and only if satisfies the following property: if the set $\{T < \gamma\}$ is not empty, then there is a number r such that $\{T < \gamma\} = \{T = r < \gamma\} = \{r < \gamma\}$.

Further properties

Proposition

(i) *A process* $X = (X_t)_{t \in \mathbb{R}_+}$ *is optional if and only if it is measurable* (*wrt* $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+)$) *and there is a deterministic function* $F(t)$, $0 \leqslant t < t_G$, *such that* $X_t = F(t)$ *on* $\{t < \gamma \wedge t_G\}$.

(ii) *A process* $X = (X_t)_{t \in \mathbb{R}_+}$ *is predictable if and only if it is measurable and there is a deterministic function* $F(t)$, $t \in \mathfrak{T}$, *such that* $X_t = F(t)$ *on* $\{t \leqslant \gamma\} \cap \{t \in \mathfrak{T}\}$.

Proposition

Every semimartingale is a process with finite variation.

The last statement is not surprising. It was proved in Jacod & Skorohod (1994) that all semimartingales are processes with finite variation in the case of a jumping filtration. A single jump filtration is a special case of a jumping filtration.

Let $M = (M_t)_{t\in\mathbb{R}_+}$ be a uniformly integrable martingale. We assume that it is defined only up to a modification. Let M_∞ be the limit in $L^1(\mathscr{F}_\infty)$ of M_t as $t \to \infty$. Since $\{t < \gamma\}$ is an atom of $\mathscr{F}_t$ and has a positive probability for $t < t_G$, we obtain from the martingale property that $M_t = F(t)$ for all $\omega \in \{t < \gamma\}$, where

$$F(t) = \overline{G}(t)^{-1}\mathsf{E}\big(M_\infty \mathbb{1}_{\{t<\gamma\}}\big), \quad t < t_G.$$

For definiteness, we put $F(t) = 0$ for $t \geqslant t_G$. It is easy to see that the function $F(t)$ is right-continuous at every $t \geqslant 0$ and has finite left-hand limits at every $t > 0$ except maybe $t = t_G$ if $\mathsf{P}(\gamma = t_G) = 0$.

Uniformly integrable martingales (continuation)

On the other hand, fix a random variable L as an arbitrary version of $M_\infty \mathbb{1}_{\{\gamma<\infty\}}$. Then $L\mathbb{1}_{\{\gamma\leqslant t\}} = M_\infty \mathbb{1}_{\{\gamma\leqslant t\}}$ is an $\mathscr{F}_t$-measurable, and hence $M_t \mathbb{1}_{\{\gamma\leqslant t\}} = \mathsf{E}(M_\infty \mathbb{1}_{\{\gamma\leqslant t\}}|\mathscr{F}_t) = L\mathbb{1}_{\{\gamma\leqslant t\}}$. Thus we have obtained the representation

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}$$

for every $t \geqslant 0$. As we explained, trajectories of the process on the right are right-continuous and have finite left-hand limits except maybe $\tau(\omega)$, where

$$\tau(\omega) = \begin{cases} t_G, & \text{if } \mathsf{P}(\gamma = t_G) = 0 \text{ and } \gamma(\omega) \geqslant t_G; \\ +\infty, & \text{otherwise.} \end{cases}$$

It can be considered as a regular version of M.

Our goals

As we have just shown, every uniformly integrable martingale in our model has the form

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}.$$

It is easy to show from the definitions that local martingales and, more generally, σ-martingales also have this form. The difference is in the assumptions imposed on the joint distribution of γ and L and its connection with the function F.

We will see that in order M_t be a σ-martingale, it is necessary that $\mathrm{E}\big(|L|\big|\gamma\big) < \infty$. Thus we can write

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + (H(\gamma) + \widehat{L})\mathbb{1}_{\{t\geqslant\gamma\}}, \quad \mathrm{E}(\widehat{L}|\gamma) = 0.$$

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Herdegen & Herrmann (2016)

Roughly speaking, Dellacherie (1970) and Chou & Meyer (1975) consider the question how to express F via G and H. Herdegen & Herrmann (2016) solve the problem how to determine H given F and G. However, they do not give a full description of the class of local martingales in this model. In particular, they make the assumption that F is "locally" absolutely continuous with respect to G. In fact, as we will see. this condition is necessary.

Characterization and description of martingales, local martingales and σ-martingales. Canonical decomposition of special semimartingales

Theorem

Let $F(t)$, $0 \leqslant t < t_G$, be a deterministic càdlàg function, L a random variable, and a process $M = (M_t)_{t \in \mathbb{R}_+}$ be given by

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t \geqslant \gamma\}}.$$

The following statements are equivalent:

(i) $M = (M_t)_{t \in \mathbb{R}_+}$ *is a local martingale.*

(ii) $(M_t)_{t \in \mathcal{T}}$ *is a martingale.*

(iii)
$$\mathsf{E}\left[|L|\mathbb{1}_{\{\gamma \leqslant t\}}\right] < \infty, \quad t \in \mathcal{T},$$
and
$$\mathsf{E}M_t = \mathsf{E}M_0, \quad t \in \mathcal{T}.$$

Proof of (ii)⇔(iii)

In the case where $\mathscr{F} = \sigma\{\gamma\}$, equivalence (i)⇔(ii) is proved by Chou & Meyer (1975).
(ii)⇒(iii) is trivial. Let (iii) hold. The process $(M_t)_{t\in\mathfrak{T}}$ is adapted and integrable due to the first part of (iii). Moreover,

$$M_t - M_s = 0 \quad \text{on} \quad \{s \geqslant \gamma\},$$

where $0 \leqslant s < t \in \mathfrak{T}$. Hence,

$$\mathsf{E}[M_t - M_s | \mathscr{F}_s] = 0 \quad \text{on} \quad \{s \geqslant \gamma\}.$$

But $\mathsf{E}[M_t - M_s | \mathscr{F}_s]$ is $\mathscr{F}_s$-measurable and, thus, equals a constant on $\{s < \gamma\}$. And this constant must be zero since $\mathsf{E}[M_t - M_s] = 0$ by the second part of (iii).

Additional notation

Let $L^1_{\mathrm{loc}}(dG)$ be the set of all Borel functions z on $\mathcal{T}$ such that

$$\int_{[0,t]} |z(s)|\, dG(s) < \infty \quad \text{for all } t \in \mathcal{T}.$$

Given a function $Z\colon [0, t_G) \to \mathbb{R}$, let us write $Z \overset{\mathrm{loc}}{\ll} G$ if there is $z \in L^1_{\mathrm{loc}}(dG)$ such that $Z(t) = Z(0) + \int_{(0,t]} z(s)\, dG(s)$ for all $t < t_G$; in this case we put $\frac{dZ}{dG}(t) := z(t)$ for $0 < t < t_G$. Let us emphasize that in Case B this definition implies that z is dG-integrable over $[0, t_G]$ and, hence, the function Z has a finite variation over $[0, t_G)$ and there is a finite limit $\lim_{t \uparrow\uparrow t_G} Z(t) = Z(0) + \int_{(0,t_G)} z(s)\, dG(s)$. Note also that in this definition the value $z(0)$ can be chosen arbitrarily even if $G(0) > 0$; the same refers to the value $z(t_G)$ in Case B. Correspondingly, dZ/dG is defined only for $0 < t < t_G$.

Two cases

In what follows we will often distinguish between two cases:

Case A $\mathrm{P}(\gamma = t_G < \infty) = 0$ or, equivalently, $\mathcal{T} = [0, t_G)$.

Case B $\mathrm{P}(\gamma = t_G < \infty) > 0$ or, equivalently, $\mathcal{T} = [0, t_G]$.

Main condition

Let G be a distribution function of a law on $[0, +\infty]$. We will say that the triple (F, G, H) satisfy the main condition if

- $F\colon [0, t_G) \to \mathbb{R}$, $F \overset{\mathrm{loc}}{\ll} G$, $H\colon \mathcal{T} \to \mathbb{R}$, $H \in L^1_{\mathrm{loc}}(dG)$.
- $$F(t)\overline{G}(t) + \int\limits_{(0,t]} H(s)\, dG(s) = F(0)\overline{G}(0), \quad t < t_G.$$
- (only in Case B)
$$\lim_{t \uparrow\uparrow t_G} F(t) = H(t_G).$$

Proposition

Let $H \in L^1_{\mathrm{loc}}(dG)$. Define

$$F(t) = \overline{G}(t)^{-1}\Big[F(0)\overline{G}(0) - \int_{(0,t]} H(s)\,dG(s)\Big], \quad 0 < t < t_G,$$

where $F(0)$ is an arbitrary real number in Case A and

$$F(0) = \overline{G}(0)^{-1}\int_{(0,t_G]} H(s)\,dG(s)$$

in Case B. Then $F \overset{\mathrm{loc}}{\ll} G$ and the main condition holds.

Conversely, if the main condition holds, then F satisfies the above relations.

Two propositions

Proposition

Let $F \overset{\text{loc}}{\ll} G$. *Define* $H(0)$ *arbitrarily,*

$$H(t) = F(t) - \overline{G}(t-)\frac{dF}{dG}(t), \quad 0 < t < t_G,$$

and $H(t_G) = \lim_{t \uparrow\uparrow t_G} F(t)$ *in Case B. Then* $H \in L^1_{\text{loc}}(dG)$ *and the main condition holds.*

Conversely, if the main condition holds, then H satisfies the above relations.

Comment

The second proposition is essentially due to Herdegen & Herrmann (2016). They write the expression for H in an equivalent form:

$$H(t) = F(t-) - \overline{G}(t)\frac{dF}{dG}(t), \quad 0 < t < t_G.$$

Theorem

In order that a right-continuous process $M = (M_t)_{t\in\mathbb{R}_+}$ be a σ-martingale it is necessary and sufficient that there are a triple (F, G, H) satisfying the main condition and a random variable $\widehat{L}$ satisfying

$$\mathsf{E}\big(|\widehat{L}|\big|\gamma\big) < \infty \qquad \textit{and} \qquad \mathsf{E}[\widehat{L}|\gamma] = 0$$

such that

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + \big(H(\gamma) + \widehat{L}\big)\mathbb{1}_{\{t\geqslant\gamma\}}, \quad t \in \mathbb{R}_+.$$

Theorem

In order that a right-continuous process $M = (M_t)_{t \in \mathbb{R}_+}$ *be a local martingale it is necessary and sufficient that there are a triple* (F, G, H) *satisfying the main condition and a random variable* $\widehat{L}$ *satisfying*

$$\mathsf{E}\big(|\widehat{L}|\mathbb{1}_{\{\gamma \leqslant t\}}\big) < \infty, \quad t \in \mathcal{T}, \qquad \text{and} \qquad \mathsf{E}[\widehat{L}|\gamma] = 0,$$

such that

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + \big(H(\gamma) + \widehat{L}\big)\mathbb{1}_{\{t \geqslant \gamma\}}, \quad t \in \mathbb{R}_+.$$

Herdegen & Herrmann (2016) prove that, in Case B, if H is not dG-integrable over $(0, t_G)$, then M defined as above is not a semimartingale. We add that, also in Case B, if H is dG-integrable over $(0, t_G)$, $F(t)$ satisfies the equality required in the corresponding propositioncorresponding condition but $F(0)$ is not as prescribed, the process M constructed in the same way, is a supermartingale or a submartingale, but not a local martingale. The fact that $H(0)$ can be chosen arbitrarily in the second proposition says only that L can be an arbitrary integrable random variable on the set $\{\gamma = 0\}$, which is evident ab initio. On the contrary, the fact that $F(0)$ can be chosen arbitrarily in the first proposition in Case A is an interesting feature of this model. It says that, given the terminal value M_∞ of M (on $\{\gamma < \infty\}$), one can freely choose the initial value M_0 of M (on $\{\gamma > 0\}$) to keep the property of being a local martingale for M.

Classification

Let us say that a local martingale $M = (M_t)_{t \in \mathbb{R}_+}$ has

type 1 if the limit $M_\infty = \lim_{t\to\infty} M_t$ does not exist with positive probability or exists with probability one but not integrable: $\mathrm{E}|M_\infty| = \infty$;

type 2a if M is a closed supermartingale (in particular, $\mathrm{E}|M_\infty| < \infty$) and $\mathrm{E}M_\infty < \mathrm{E}M_0$;

type 2b if M is a closed submartingale (in particular, $\mathrm{E}|M_\infty| < \infty$) and $\mathrm{E}M_\infty > \mathrm{E}M_0$;

type 3 if M is a uniformly integrable martingale (in particular, $\mathrm{E}|M_\infty| < \infty$ and $\mathrm{E}M_\infty = \mathrm{E}M_0$) and $\mathrm{E}\sup_t |M_t| = \infty$;

type 4 if M has an integrable variation: $\mathrm{E}\,\mathrm{Var}(M)_\infty < \infty$.

Classification

The next theorem complements the classification of the limit behaviour of local martingales that was considered in Herdegen & Herrmann (2016) in the case where $\mathscr{F} = \sigma\{\gamma\}$.

Theorem

Let $M = (M_t)_{t \in \mathbb{R}_+}$ be a local martingale with a representation

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + (H(\gamma) + \widehat{L})\mathbb{1}_{\{t \geqslant \gamma\}}, \quad t \in \mathbb{R}_+.$$

Then in Case B the local martingale M has type 4. In Case A all types are possible. Namely,

(i) *M has type 1 if and only if $\mathsf{E}\big(|\widehat{L}|\mathbb{1}_{\{\gamma<\infty\}}\big) = \infty$ or $\int_{[0,t_G)} |H(s)|\, dG(s) = \infty$.*

(ii) *If $\mathsf{P}(\gamma = \infty) > 0$, $\mathsf{E}\big(|\widehat{L}|\mathbb{1}_{\{\gamma<\infty\}}\big) < \infty$, and $\int_{\mathbb{R}_+} |H(s)|\, dG(s) < \infty$ then M has type 4.*

Theorem (continued)

(iii) *If* $\mathsf{P}(\gamma = \infty) = 0$, $\mathsf{E}|\widehat{L}| < \infty$, *and* $\int_{[0,t_G)} |H(s)|\, dG(s) < \infty$ *then*

(iii.i) *M has type* 2a (*resp.*, 2b) *if and only if* $\lim_{t \uparrow\uparrow t_G} F(t)\overline{G}(t) > 0$ (*resp.*, $\lim_{t \uparrow\uparrow t_G} F(t)\overline{G}(t) < 0$);

(iii.ii) *M has type* 3 *if and only if*

$$\lim_{t \uparrow\uparrow t_G} F(t)\overline{G}(t) = 0 \quad \textit{and} \quad \int_{[0,t_G)} \overline{G}(s) \left| \frac{dF}{dG}(s) \right| dG(s) = \infty;$$

(iii.iii) *M has type* 4 *if and only if*

$$\int_{[0,t_G)} \overline{G}(s) \left| \frac{dF}{dG}(s) \right| dG(s) < \infty \tag{6}$$

Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a semimartingale, $X_0 = 0$. Then we can write

$$X_t = F(t \wedge \gamma) + \big(X_\gamma - F(\gamma)\big)\mathbb{1}_{\{t \geqslant \gamma\}} + (X_t - X_\gamma)\mathbb{1}_{\{t > \gamma\}},$$

F is a deterministic function on $\mathcal{T}$. The first and third terms on the right are predictable, and the compensator of the second one, if it exists, has a form $R(t \wedge \gamma)$, where R is a deterministic function on $\mathcal{T}$. It follows that X is a special semimartingale if and only if

$$\mathsf{E}\big[|X_\gamma|\mathbb{1}_{\{\gamma \leqslant t\}}\big] < \infty, \quad t \in \mathcal{T}.$$

In this case we put

$$K(t) = \mathsf{E}[X_\gamma | \gamma = t] - F(t), \quad t \in \mathcal{T}.$$

Canonical decomposition of semimartingales

It follows from the description of local martingales that

$$K(t) = \overline{G}(t-)\frac{dR}{dG}(t), \quad 0 < t < t_G,$$

and $K(t_G) - R(t_G) = \lim_{t \uparrow\uparrow t_G} -R(t)$ in Case B. Thus, we obtain

$$R(t) = \int_{(0,t]} \overline{G}(s-)^{-1} K(s)\, dG(s), \quad 0 \leqslant t < t_G,$$

and

$$R(t_G) = \int_{(0,t_G)} \overline{G}(s-)^{-1} K(s)\, dG(s) + K(t_G)$$

$$= \int_{(0,t_G]} \overline{G}(s-)^{-1} K(s)\, dG(s) \qquad \text{in Case B.}$$

Thank you for your attention

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