Random Mappings with Constraints on the Component Sizes

A.L. Yakymiv*
Steklov Mathematical Institute RAS,
8 Gubkina St., Moscow 119991, Russia
arsen@mi-ras.ru

Let X be an arbitrary fixed finite set having n elements. By $\mathfrak{S}_n$ we denote a semigroup of all mappings from the set X into itself. Each mapping $\omega(\cdot) \in \mathfrak{S}_n$ corresponds to a graph $\Gamma(X, \sigma)$ whose vertices $x, y \in X$ are connected by an arc (x,y) if $y = \omega(x)$. As it's known, every graph $\Gamma(X, \omega)$ consists of connected components, each consisting of a single cycle and trees. Fix an arbitrary set D of natural numbers. By $\mathfrak{S}_n(D)$ denote a set of mappings from $\mathfrak{S}_n$ with component sizes belonging to the set D. Let a random mapping $\sigma = \sigma_n(D)$ be uniformly distributed on $\mathfrak{S}_n(D)$. This mapping is introduced in [1]. Random mappings without restrictions have mach longer history. See, for example [2, 3].

Let as consider the following two classes of the sets D. We say that a set D belongs to the class F_1 iff $D = \bigcup_{i=1}^{M} D_i$, where $M \in N,\ D_i = \{m \in N :\ m = a_i k + b_i,\ k = 0, 1, 2, \dots\}$ and the integers $a_i > 1,\ 1 \le b_i \le a_i - 1,\ (a_i, b_i) = 1$ with $D_i \cap D_j = \emptyset,\ \forall i \ne j$. Also, we say that a set D belongs to the class F_2, iff $D = \{m \in N :\ m/k_i \notin N,\ i = 1, \dots, s\}$ for some $s \in N$ and $k_1, \dots, k_s \in N$ such that $k_i \ge 2,\ i = 1, \dots, s$ and $(k_i, k_j) = 1\ \forall i \ne j$.

Fore $x \in (0, 1)$ and $\gamma \in R^1$, denote

$$f_\gamma(x) = (1 - x)^{-\gamma},$$

Also put

$$l(n) = \sum_{i \in D(n)} \frac{1}{i}.$$

By $\varrho = \varrho(D)$, denote the density of the set D in the set of natural numbers, and put $\alpha = \varrho/2,\ D(n) = D \cap [1, n]$.

Theorem 1 *Let $D \in F_1 \cup F_2$. Then, for some $\beta \in (0, 1/2]$*

$$\mathsf{E}\zeta_n = \frac{1}{2} l(n) + J(n) + O\left(\frac{1}{n^\alpha}\right) + O\left(\frac{\ln n}{n^\beta}\right)$$

*This work is supported by the Russian Science Foundation under grant 19-11-00111

as $n \to \infty$, *where*

$$J(n) = \frac{1}{2} \sum_{m \in D(n-1)} \frac{1}{m} \left(1 - f_{1-\varrho}\left(\frac{m}{n}\right)\right) \to \frac{\varrho}{2} \int_0^1 \frac{1 - f_{1-\varrho}(x)}{x}\, dx.$$

Theorem 2 *Let* $D \in F_1 \cup F_2$. *Then*

$$\mathsf{Var}(\zeta_n) = \mathsf{E}\zeta_n + \frac{1}{4} J_0(n) + \frac{1}{4} J_1(n) + O(n^{-\alpha} \ln n) + O(n^{-\beta} \ln^2 n) + O(n^{-\nu}),$$

where

$$J_0 = \sum_{m,k \in D, m+k \geq n} \frac{1}{mk} f_\alpha\left(\frac{m}{n}\right) f_\alpha\left(\frac{k}{n}\right)$$

$$\to \varrho^2 \int_{(x,y)\in(0,1)^2 : x+y \geq 1} (1-x)^{-\alpha} (1-y)^{-\alpha} \frac{dx\, dy}{xy} < \infty,$$

$$J_1 = \sum_{m,k \in D(n-1),\ m+k<n} \frac{1}{mk} \left(f_\alpha\left(\frac{m+k}{n}\right) - f_\alpha\left(\frac{m+k}{n} - \frac{mk}{n^2}\right)\right).$$

$$\to \varrho^2 \int_{x,y>0,\ x+y<1} \frac{1}{xy} (f_\alpha(x+y) - f_\alpha(x+y-xy))\, dx\, dy < \infty,$$

$$\nu = \frac{1-\alpha}{2(2-\alpha)}.$$

In this talk, we also give a survay on the asymptotic results obtained earlier for $\sigma_n(D)$.

References

[1] A.N. Timashëv. Random components in the general allocation sceme. Moscow, Publishing House "Academy", 2017 (in Russian).

[2] B. Harris. A survay of the early history of random mappings, Probabilistic Methods in Discrete Mathematics, TVP, Moscow, 1993, pp. 1–22.

[3] Kolchin, V. F. Random Mappings, Optimization Software, New York, 1986.