On periodic branching random walks on $\mathbb{Z}^d$ with heavy tails

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We consider a branching random walk (BRW) with periodically located branching sources and a periodic matrix of transition intensities. The conditions imposed on the transition intensities imply the infinite variance of jumps. Such processes usually called BRW with heavy tails. The case of BRW with a finite number of branching sources and finite variance of jumps is widely studied (see [8], [5] and reference therein). Extinction probabilities and moments behaviour in case of heavy tails are obtained in [3] and [4] correspondingly. The asymptotic behaviour of the mean number of particles in the case of periodically located branching sources and the finite variance of jumps is obtained in [6]. There are many results about heavy-tailed random walks (see [1]).

Let $g_1, \ldots, g_d \in \mathbb{Z}^d$ be a basis in $\mathbb{Z}^d$. We call the set

$$\Gamma = \Big\{ g \in \mathbb{Z}^d : g = \sum_{j=1}^{d} n_j g_j,\ n_j \in \mathbb{Z},\ j = 1, \ldots, d \Big\},$$

by a lattice. Let the transition intensities be denoted by $a(v, u)$, for $v, u \in \mathbb{Z}^d$. They always satisfy:

(i) $a(v, u) \geqslant 0, \quad v \neq u$;
(ii) $a(v, v) < 0$.

Additionally we assume the following conditions:

(iii) $\sum\limits_{u \in \mathbb{Z}^d} a(v, u) = 0$;
(iv) $a(v, u) = a(u, v) = a(v + g, u + g), \quad \forall g \in \Gamma$;

Let $x = \{x_1, x_2, \ldots, x_d\} \in \mathbb{R}^d$. We denote by $\|x\|_\infty$ the following norm

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots |x_d|\}.$$

Let the matrix P be defined by the equalities

$$g_j = P e_j, \quad j = 1, \ldots, d.$$

(v) There is $\alpha \in (0, 2)$ such that

$$a(v_j + g, v_k) \|P^{-1} g\|_\infty^{d+\alpha} \to h_{jk}, \text{ for } \|g\|_\infty \to +\infty,$$

where $h_{jk} \in [0, +\infty)$, for all $j, k = 1, \ldots, d$ and at least one of them is strictly positive.

(vi) The graph $G = (\mathbb{Z}^d, \mathcal{E})$ with the vertex set $\mathbb{Z}^d$ and the edge set

$$\mathcal{E} = \{(v, u) : a(v, u) > 0,\ v, u \in \mathbb{Z}^d\}$$

is connected.

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(vii) For all $v, u \in \mathbb{Z}^d$ and any $g \in \Gamma$
$$a(v, u-g) = a(v, u+g).$$

The condition (iii) means that the Markov process of random walks is conservative. The transition intensities matrix is symmetric and periodic with respect to Γ from (iv). It also satisfies some additional symmetry condition (vii). The condition (v) characterizes the behaviour of transition intensities for large jumps. It follows from (vi) that a random walk is irreducible.

Let the branching intensity to $k \in \mathbb{N} \cup \{0\}$ offspings in a vertex $v \in \mathbb{Z}^d$ be denoted by $b_k(v)$. Then:

(a) $b_k(v) \geqslant 0$ for $k \neq 1$,
(b) $b_1(v) \leqslant 0$.

Additionally we assume that:

(c) $\sum\limits_{k=0}^{+\infty} b_k(v) = 0$;
(d) $\beta_1(v) = \sum\limits_{k=1}^{+\infty} k b_k(v) < \infty$;
(e) $\beta_1(v+g) = \beta_1(v), \quad g \in \Gamma$.

Condition (c) means that Markov process of branching is conservative. It follows from (d) that the mean number of offsprings from one branching is finite.

Suppose that at time $t = 0$ there is only one particle at a vertex $v \in \mathbb{Z}^d$. We denote by $M(v, u, t)$ the mean number of particles in a vertex $u \in \mathbb{Z}^d$ at time t. Let p be a number of vertices $v \in \mathbb{Z}^d$ which can not be obtained one from another by translations on vectors from Γ. We denote by A the following matrix

$$A = \begin{pmatrix} \widetilde{a}_{11} + \beta_1 & \widetilde{a}_{12} & \cdots & \widetilde{a}_{1p} \\ \widetilde{a}_{21} & \widetilde{a}_{22} + \beta_2 & \cdots & \widetilde{a}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_{p1} & \widetilde{a}_{p2} & \cdots & \widetilde{a}_{pp} + \beta_p \end{pmatrix},$$

with entities $\widetilde{a}_{jk}$ defined by
$$\widetilde{a}_{jk} = \sum_{g \in \Gamma} a(v_j + g, v_k).$$

Denote by λ_1 the leading eigenvalue of A.

Theorem 1. *Let a BRW satisfy conditions* (i – vii) *and* (a – e). *Then following asymptotic relation holds for large t*
$$M(v, u, t) = m(v, u) e^{\lambda_1 t} t^{-\frac{d}{\alpha}} (1 + o(1)),$$
where $\alpha \in (0, 2)$ is defined in condition (v) *and $m(v, u)$ can be computed explicitly.*

In this work we use the results of [2] and [7] extensively.

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