

On supercritical branching random walks on periodic lattices

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We consider a model of a continuous-time branching random walk on the lattice \mathbb{Z}^d . We assume that particles evolve independently of each other and the branching sources are distributed periodically.

Let us describe the model in more details. Let g_1, \dots, g_d be a family of linearly independent (not necessarily orthogonal) vectors with integer coordinates. By a lattice we mean a set

$$\Gamma = \left\{ g \in \mathbb{Z}^d : g = \sum_{j=1}^d n_j g_j, n_j \in \mathbb{Z}, j = 1, \dots, d \right\}.$$

We denote by $\|g\|$ the Euclidean norm of a vector from \mathbb{R}^d . We introduce the equivalence relation on \mathbb{Z}^d as follows: points $u, v \in \mathbb{Z}^d$ are said to be equivalent if $u - v \in \Gamma$. The corresponding quotient space can always be identified with some set $\{v_1, \dots, v_p\}$ of pairwise nonequivalent elements of \mathbb{Z}^d .

We assume that the random walk of particles is governed by a matrix of transition intensities $(a(v, u))_{v, u \in \mathbb{Z}^d}$ which features the following properties:

- (i) $a(v, u) \geq 0, v \neq u, a(v, v) < 0$;
- (ii) $\sum_{u \in \mathbb{Z}^d} a(v, u) = 0$;
- (iii) $a(v, u) = a(u, v) = a(v + g, u + g), \forall g \in \Gamma$;
- (iv) $\sum_{u \in \mathbb{Z}^d} \|u\|^2 |a(v, u)| < +\infty, v \in \mathbb{Z}^d$;
- (v) for any $v, u \in \mathbb{Z}^d$ there exists a path $v = u_0, u_1, \dots, u_m = u$ such that $a(u_{i-1}, u_i) > 0, i = 1, \dots, m$.

The branching source at a vertex $v \in \mathbb{Z}^d$ is described by a generating function $B(v, s) = \sum_{k=0}^{\infty} b_k(v) s^k$ where the coefficients $b_k(v), k \in \mathbb{N} \cup \{0\}$, satisfy the conditions:

- (a) $b_k(v) \geq 0, k \neq 1, b_1(v) \leq 0$;
- (b) $\sum_{k=0}^{+\infty} b_k(v) = 0$;
- (c) $\beta_n(v) = B^{(n)}(v, 1) = \sum_{k=n}^{+\infty} k(k-1) \dots (k-n+1) b_k(v) < \infty, n \in \mathbb{N}$;
- (d) $\beta_n(v+g) = \beta_n(v), g \in \Gamma, n \in \mathbb{N}$.

We assume that initially at $t = 0$ there is one particle at a point $v \in \mathbb{Z}^d$. Each particle, which at time t is located at some point v , can, independently of the remaining particles in the system, in time $[t, t+h)$ transit with probability $p(h, v, u) = a(v, u)h + o(h)$ to a point $u \neq v$ or generate $k \neq 1$ descendants located at the point v with probability $p_k(h, v) = b_k(v)h + o(h)$ (for $k = 0$ we assume that

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the number of descendants is 0; that is, the particle dies) or remain unchanged (that is, no changes occur) with probability

$$1 - \sum_{u \neq v} a(v, u)h - \sum_{k \neq 1} b_k(v)h + o(h).$$

The main object of interest will be the long-time asymptotic behaviour of the moments of local particle number at an arbitrary point. Let the moment of order n of local particle number at an arbitrary point $u \in \mathbb{Z}^d$ be denoted by $M_n(v, u, t)$. It is easy to check that $M_n(v, u, t)$ is a solution to the Cauchy problem

$$\begin{cases} \partial_t M_n(v, u, t) &= (\mathcal{A}M_n)(v, u, t) + R_n(v, u, t), \\ M_n(v, u, 0) &= \delta_u(v), \end{cases}$$

where

$$\mathcal{A}f(v) = \sum_{u \in \mathbb{Z}^d} a(v, u)f(u) + \beta_1(v)f(v),$$

and

$$R_n(v, u, t) = \begin{cases} \sum_{r=2}^n \frac{\beta_r(v)}{r!} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = n}} \frac{n!}{i_1! \dots i_r!} M_{i_1}(v, u, t) \dots M_{i_r}(v, u, t), & n \geq 2, \\ 0, & n = 1. \end{cases}$$

In [1] it was shown that the asymptotic behaviour of the mean number of particles at an arbitrary point of the lattice ($M_1(v, u, t)$) depended on the right edge of spectrum of the operator \mathcal{A} . It was also shown that the right edge of the spectrum of \mathcal{A} coincided with the leading eigenvalue (λ_1) of the matrix

$$\begin{pmatrix} \tilde{a}_{11} + \beta_1(v_1) & \tilde{a}_{12} & \dots & \tilde{a}_{1p} \\ \tilde{a}_{21} & \tilde{a}_{22} + \beta_1(v_2) & \dots & \tilde{a}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{p1} & \tilde{a}_{p2} & \dots & \tilde{a}_{pp} + \beta_1(v_p) \end{pmatrix},$$

where

$$\tilde{a}_{jk} = \sum_{g \in \Gamma} a(v_j + g, v_k).$$

A branching random walk is called supercritical if the leading eigenvalue λ_1 is greater than 0.

Theorem 1. *Let a supercritical branching random walk satisfy the conditions (i–v) and $(a - d)$. Then as $t \rightarrow \infty$*

$$\ln M_n(v, u, t) = n\lambda_1 t - \frac{dn}{2} \ln t + O(1).$$

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СПИСОК ЛИТЕРАТУРЫ

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