On supercritical branching random walks on periodic lattices

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We consider a model of a continuous-time branching random walk on the lattice $\mathbb{Z}^d$. We assume that particles evolve independently of each other and the branching sources are distributed periodically.

Let us describe the model in more details. Let $g_1, \dots, g_d$ be a family of linearly independent (not necessarily orthogonal) vectors with integer coordinates. By a lattice we mean a set

$$\Gamma = \Big\{ g \in \mathbb{Z}^d : g = \sum_{j=1}^{d} n_j g_j,\ n_j \in \mathbb{Z},\ j = 1, \dots, d \Big\}.$$

We denote by $\|g\|$ the Euclidean norm of a vector from $\mathbb{R}^d$. We introduce the equivalence relation on $\mathbb{Z}^d$ as follows: points $u, v \in \mathbb{Z}^d$ are said to be equivalent if $u - v \in \Gamma$. The corresponding quotient space can always be identified with some set $\{v_1, \dots, v_p\}$ of pairwise nonequivalent elements of $\mathbb{Z}^d$.

We assume that the random walk of particles is governed by a matrix of transition intensities $\big(a(v,u)\big)_{v,u \in \mathbb{Z}^d}$ which features the following properties:

(i) $a(v,u) \geqslant 0, \quad v \neq u, \quad a(v,v) < 0;$
(ii) $\sum\limits_{u \in \mathbb{Z}^d} a(v,u) = 0;$
(iii) $a(v,u) = a(u,v) = a(v+g, u+g), \quad \forall g \in \Gamma;$
(iv) $\sum\limits_{u \in \mathbb{Z}^d} \|u\|^2 |a(v,u)| < +\infty,\ v \in \mathbb{Z}^d;$
(v) for any $v, u \in \mathbb{Z}^d$ there exists a path $v = u_0, u_1, \dots, u_m = u$ such that $a(u_{i-1}, u_i) > 0,\ i = 1, \dots, m.$

The branching source at a vertex $v \in \mathbb{Z}^d$ is described by a generating function $B(v,s) = \sum\limits_{k=0}^{\infty} b_k(v) s^k$ where the coefficients $b_k(v),\ k \in \mathbb{N} \cup \{0\}$, satisfy the conditions:

(a) $b_k(v) \geqslant 0, \quad k \neq 1, \quad b_1(v) \leqslant 0;$
(b) $\sum\limits_{k=0}^{+\infty} b_k(v) = 0;$
(c) $\beta_n(v) = B^{(n)}(v,1) = \sum\limits_{k=n}^{+\infty} k(k-1)\dots(k-n+1) b_k(v) < \infty, \quad n \in \mathbb{N};$
(d) $\beta_n(v+g) = \beta_n(v), \quad g \in \Gamma, \quad n \in \mathbb{N}.$

We assume that initially at $t = 0$ there is one particle at a point $v \in \mathbb{Z}^d$. Each particle, which at time t is located at some point v, can, independently of the remaining particles in the system, in time $[t, t+h)$ transit with probability $p(h,v,u) = a(v,u)h + o(h)$ to a point $u \neq v$ or generate $k \neq 1$ descendants located at the point v with probability $p_k(h,v) = b_k(v)h + o(h)$ (for $k = 0$ we assume that

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the number of descendants is 0; that is, the particle dies) or remain unchanged (that is, no changes occur) with probability

$$1-\sum_{u\neq v}a(v,u)h-\sum_{k\neq 1}b_k(v)h+o(h).$$

The main object of interest will be the long-time asymptotic behaviour of the moments of local particle number at an arbitrary point. Let the moment of order n of local particle number at an arbitrary point $u\in\mathbb{Z}^d$ be denoted by $M_n(v,u,t)$. It is easy to check that $M_n(v,u,t)$ is a solution to the Cauchy problem

$$\begin{cases}\partial_t M_n(v,u,t) &= \big(\mathcal{A}M_n\big)(v,u,t)+R_n(v,u,t),\\ M_n(v,u,0) &= \delta_u(v),\end{cases}$$

where

$$\mathcal{A}f(v)=\sum_{u\in\mathbb{Z}^d}a(v,u)f(u)+\beta_1(v)f(v),$$

and

$$R_n(v,u,t)=\begin{cases}\sum\limits_{r=2}^{n}\frac{\beta_r(v)}{r!}\sum\limits_{\substack{i_1,\dots,i_r>0\\ i_1+\ldots+i_r=n}}\frac{n!}{i_1!\ldots i_r!}M_{i_1}(v,u,t)\ldots M_{i_r}(v,u,t), & n\geq 2,\\ 0, & n=1.\end{cases}$$

In [1] it was shown that the asymptotic behaviour of the mean number of particles at an arbitrary point of the lattice ($M_1(v,u,t)$) depended on the right edge of spectrum of the operator $\mathcal{A}$. It was also shown that the right edge of the spectrum of $\mathcal{A}$ coincided with the leading eigenvalue (λ_1) of the matrix

$$\begin{pmatrix}\widetilde{a}_{11}+\beta_1(v_1) & \widetilde{a}_{12} & \cdots & \widetilde{a}_{1p}\\ \widetilde{a}_{21} & \widetilde{a}_{22}+\beta_1(v_2) & \cdots & \widetilde{a}_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \widetilde{a}_{p1} & \widetilde{a}_{p2} & \cdots & \widetilde{a}_{pp}+\beta_1(v_p)\end{pmatrix},$$

where

$$\widetilde{a}_{jk}=\sum_{g\in\Gamma}a(v_j+g,v_k).$$

A branching random walk is called supercritical if the leading eigenvalue λ_1 is greater than 0.

Theorem 1. *Let a supercritical branching random walk satisfies the conditions* (i–v) *and* $(a-d)$*. Then as* $t\to\infty$

$$\ln M_n(v,u,t)=n\lambda_1 t-\frac{dn}{2}\ln t+O(1).$$

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Список литературы

[1] M. V. Platonova, K. S. Ryadovkin. Branching random walks on $\mathbb{Z}^d$ with periodic branching sources. *Theory Probab. Appl.*, 64:2 (2019), 229–248.

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