Investigation of a Stochastic Model of Nonlinear Filtration

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Consider a complete probability space $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ and the set of real numbers $\mathbf{R}$ endowed with a Borel σ-algebra. According to [1], a measurable mapping $\xi : \Omega \to \mathbf{R}$ is called a *random variable*. The set of random variables having zero expectations (i.e. $\mathbf{E}\xi = 0$) and finite variances (i.e. $\mathbf{D}\xi < +\infty$) forms Hilbert space $\mathbf{L_2}$ with the scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$, where $\mathbf{E}$, $\mathbf{D}$ is the expectation and variance of the random variable, respectively.

Let $\mathcal{I} \subset \mathbf{R}$ be a set. Consider two mappings: $f : \mathcal{I} \to \mathbf{L_2}$, that each $t \in \mathcal{I}$, associates with a random variable $\xi \in \mathbf{L_2}$, and $g : \mathbf{L_2} \times \Omega \to \mathbf{R}$, that each pair (ξ, ω) associates with a point $\xi(\omega) \in \mathbf{R}$. A mapping $\eta : \mathcal{I} \times \Omega \to \mathbf{R}$ of the form $\eta = \eta(t, \omega) = g(f(t), \omega)$ is called an (*one-dimensional*) *random process*. According to [1], a random process η is called *continuous*, if almost surely all its trajectories are continuous. Denote by $\mathbf{CL_2}$ the set of continuous random processes, which forms a Banach space. Fix $\eta \in \mathbf{CL_2}$ and $t \in \mathcal{I}$ and denote by $\mathcal{N}_t^\eta$ the σ-algebra generated by the random variable $\eta(t)$. Denote $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$.

Definition 1. (i) Suppose that $\eta \in \mathbf{CL_2}$. The derivative

$$\overset{o}{\eta} = D_S\eta = \frac{1}{2}(D + D_*)\eta = D\eta(t,\cdot) + D_*\eta(t,\cdot) =$$

$$= \lim_{\triangle t \to 0+} E_t^\eta \left(\frac{\eta(t + \triangle t, \cdot) - \eta(t, \cdot)}{\triangle t} \right) + \lim_{\triangle t \to 0+} E_t^\eta \left(\frac{\eta(t, \cdot) - \eta(t - \triangle t, \cdot)}{\triangle t} \right)$$

is called the symmetric mean derivative, where $D\eta(t, \cdot)$ is *derivative on the right* (*on the left* $D_*\eta(t, \cdot)$) *of a random process* η *at the point* $t \in (\varepsilon, \tau)$, if the limit exists in the sense of a uniform metric on $\mathbf{R}$. A random process η is called *mean differentiable on the right (on the left) on* $\mathcal{I}$, if there exists the mean derivative on the right (on the left) at each point $t \in \mathcal{I}$.

Futher, the symmetric mean derivative is called the *Nelson–Gliklikh derivative*. Denote the l-th Nelson–Gliklikh derivative of the random process η by $\overset{o}{\eta}{}^{(l)}$, $l \in \mathbf{N}$. Note that the Nelson–Gliklikh derivative coincides with the classical derivative, if $\eta(t)$ is a deterministic function.

Consider the space of "noises" $\mathbf{C}^l\mathbf{L}_2$, $l \in \mathbf{N}$, i.e. the space of random processes from $\mathbf{CL}_2$, whose trajectories are almost surely differentiable by Nelson–Gliklikh on $\mathcal{I}$ up to the order l inclusive.

Consider a real separable Hilbert space $(\mathbf{H}, < \cdot, \cdot >)$ identified with its conjugate space with orthonormal basis $\{\varphi_k\}$.Each element $u \in \mathbf{H}$ can be represented as $u = \sum_{k=1}^{\infty} < u, \varphi_k > \varphi_k$. Next, choose a monotonely decreasing numerical sequence $K = \{\mu_k\}$ such that $\sum_{k=1}^{\infty} \mu_k^2 < +\infty$. Consider a sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$, such that $\sum_{k=1}^{\infty} \mu_k^2 \mathbf{D}\xi_k < +\infty$. Denote by $\mathbf{H}_K\mathbf{L}_2$ the Hilbert space of *random K-variables* having the form $\xi = \sum_{k=1}^{\infty} \mu_k \xi_k \varphi_k$. Moreover, a random K-variable $\xi \in \mathbf{H}_K\mathbf{L}_2$ exists, if, for example, $\mathbf{D}\xi_k <$ const $\forall k$. Note that space $\mathbf{H}_K\mathbf{L}_2$ is a Hilbert space with scalar product $(\xi^1, \xi^2) = \sum_{k=1}^{\infty} \mu_k^2 \mathbf{E}\xi_k^1 \xi_k^2$. Consider a sequence of random processes $\{\eta_k\} \subset \mathbf{CL}_2$ and define $\mathbf{H}$-*valued continuous stochastic K-process*

$$\eta(t) = \sum_{k=1}^{\infty} \mu_k \eta_k(t) \varphi_k \tag{1}$$

if series (1) converges uniformly by the norm $\mathbf{H}_K\mathbf{L}_2$ on any compact set in $\mathcal{I}$. Consider the Nelson–Gliklikh derivatives of random K-process

$$\overset{o}{\eta}{}^{(l)}(t) = \sum_{k=1}^{\infty} \mu_k \overset{o}{\eta}{}_k^{(l)}(t) \varphi_k$$

on the assumption that there exist the Nelson–Gliklikh derivatives up to the order l inclusive in the right-hand side, and all series converge uniformly according to the norm $\mathbf{H}_K\mathbf{L}_2$ on any compact from $\mathcal{I}$. Next, consider the space $\mathbf{C}(\mathcal{I}; \mathbf{H}_K\mathbf{L}_2)$ of continuous stochastic K-processes and the space $\mathbf{C}^l(\mathcal{I}; \mathbf{H}_K\mathbf{L}_2)$ of stochastic K-processes whose trajectories are almost surely continuously differentiable by Nelson–Gliklikh up to the order $l \in \mathbf{N}$ inclusive.

Consider dual pairs of reflexive Banach spaces $(\mathcal{H}, \mathcal{H}^*)$ and $(\mathcal{B}, \mathcal{B}^*)$, such that embeddings

$$\mathcal{B} \hookrightarrow \mathcal{H} \hookrightarrow \mathbf{H} \hookrightarrow \mathcal{H}^* \hookrightarrow \mathcal{B}^* \tag{2}$$

are dense and continuous. Let an operator $L \in \mathcal{L}(\mathcal{H}; \mathcal{H}^*)$ be linear, continuous, self-adjoint, non-negative defined Fredholm operator, and an operator $M \in C^k(\mathcal{B}; \mathcal{B}^*), k \geq 1$, be dissipative. In space $\mathbf{H}$ choose an orthonormal basis $\{\varphi_k\}$ so that $\mathrm{span}\{\varphi_1, \varphi_2, ..., \varphi_l\} = \ker L, \dim \ker L = l$ and the following condition holds: $\{\varphi_k\} \subset \mathcal{B}$.

Taking into account that the operator L is self-adjoint and Fredholm, we identify $\mathbf{H} \supset \ker L \equiv \mathrm{coker}\, L \subset \mathbf{H}^*$ and, similarly, construct the space $\mathbf{H}_K^*\mathbf{L}_2$ according to the corresponding orthonormal basis. We use the subspace $\ker L$ in order to construct the subspace $[\ker L]_K\mathbf{L}_2 \subset \mathbf{H}_K\mathbf{L}_2$ and, similarly, the subspace $[\mathrm{coker\, L}]_{\mathrm{K}}\mathbf{L}_2 \subset \mathbf{H}_{\mathrm{K}}^*\mathbf{L}_2$. Taking into account that embeddings (2) are continuous and dense, we construct the spaces $\mathcal{H}_K^*\mathbf{L}_2 = [\mathrm{coker}\, L]_K\mathbf{L}_2 \oplus [\mathrm{im}\, L]_K\mathbf{L}_2$ and $\mathcal{B}_K^*\mathbf{L}_2 = [\mathrm{coker}\, L]_K\mathbf{L}_2 \oplus [\overline{\mathrm{im}\, L}]_K\mathbf{L}_2$.

We use the subspace $\mathrm{coim}\, L \subset \mathcal{H}$ in order to construct the subspace $[\mathrm{coim}\, L]_K\mathbf{L}_2$ such that the space $\mathcal{H}_K\mathbf{L}_2 = [\ker L]_K\mathbf{L}_2 \oplus [\mathrm{coim}\, L]_K\mathbf{L}_2$. Denote $[\ker L]_K\mathbf{L}_2 \equiv \mathcal{B}_K^0\mathbf{L}_2$ such that the space $\mathrm{coim}\, L \cap \mathcal{B}$ in order to construct the set $\mathcal{B}_K^1\mathbf{L}_2$, then $\mathcal{B}_K\mathbf{L}_2 = \mathcal{B}_K^0\mathbf{L}_2 \oplus \mathcal{B}_K^1\mathbf{L}_2$. The following lemma is correct, since the operator L is self-adjoint and Fredholm.

Lemma 1. [2] (i) Let operator $L \in \mathcal{L}(\mathcal{H}; \mathcal{H}^*)$ be a linear, continuous, self-adjoint, non-negatively defined Fredholm operator, then the operator $L \in \mathcal{L}(\mathcal{H}_K\mathbf{L}_2; \mathcal{H}_K^*\mathbf{L}_2)$, and

$$\mathbf{H}_K\mathbf{L}_2 \supset [\ker L]_K\mathbf{L}_2 \equiv [\mathrm{coker}\, L]_K\mathbf{L}_2 \subset \mathbf{H}_K^*\mathbf{L}_2$$

if

$$\mathbf{H} \supset \ker L \equiv \operatorname{coker} L \subset \mathbf{H}^*.$$

(ii) There exists a projector Q of the space $\mathcal{B}_K^*\mathbf{L}_2$ on $[\overline{\operatorname{im} L}]_K\mathbf{L}_2$ along $[\operatorname{coker} L]_K\mathbf{L}_2$.

(iii) There exists a projector P of the space $\mathcal{B}_K\mathbf{L}_2$ on $\mathcal{B}_K^1\mathbf{L}_2$ along $\mathcal{B}_K^0\mathbf{L}_2$.

Suppose that $\mathcal{I} \equiv (0, +\infty)$. We use the space $\mathbf{H}$ in order to construct the spaces of K-"noises" spaces $\mathbf{C}^k(\mathcal{I}; \mathbf{H}_K\mathbf{L}_2)$ and $\mathbf{C}^k(\mathcal{I}; \mathcal{B}_K\mathbf{L}_2),\ k \in \mathbf{N}$. Consider the stochastic Sobolev type equation

$$L \overset{o}{\eta} = M(\eta). \tag{3}$$

A solution to equation (3) is a stochastic K-process. Stochastic K-processes $\eta = \eta(t)$ and $\zeta = \zeta(t)$ are considered to be equal, if almost surely each trajectory of one of the processes coincides with a trajectory of other process.

Definition 2. A stochastic K-process $\eta \in C^1(\mathcal{I}; \mathcal{B}_K\mathbf{L}_2)$ is called a *solution to equation* (3), if almost surely all trajectories of η satisfy equation (3) for all $t \in \mathcal{I}$. A solution $\eta = \eta(t)$ to equation (3) that satisfies the initial value condition

$$\lim_{t\to 0+} (\eta(t) - \eta_0) = 0 \tag{4}$$

is called a *solution to Cauchy problem* (3), (4), if the solution satisfies condition (4) for some random K-variable $\eta_0 \in \mathcal{B}_K\mathbf{L}_2$.

Fix $\omega \in \Omega$. Let $\eta = \eta(t), t \in \mathcal{I}$ be a solution to equation (3), then η belongs to the set

$$\mathbf{M} = \begin{cases} \{\eta \in \mathcal{B}_K\mathbf{L}_2 : (\mathbf{I} - Q)M(\eta) = 0\}, & \text{if } \ker L \neq \{0\}; \\ \mathcal{B}_K\mathbf{L}_2, & \text{if } \ker L = \{0\}. \end{cases} \tag{5}$$

Theorem 1. [2] Suppose that the set $\mathbf{M}$ is a simple Banach C^k-manifold at the point $\eta_0 \in \mathbf{M}$. Then for any $\eta \in C^1(\mathcal{I}; \mathbf{M})$ exists a solution to Cauchy problem (3), (4).

Next, we consider the Dirichlet problem

$$\eta(s,t) = 0,\ (s,t) \in \partial\Omega \times \mathbf{R}_+ \tag{6}$$

for the stochastic Boussinesq equation

$$(\lambda - \Delta) \overset{o}{\eta} = \Delta(|\eta|^{p-2}\eta),\ p \geq 2. \tag{7}$$

Theorem 2. [2] Suppose that the set $\mathbf{M}$ is a simple Banach C^k-manifold at the point $\eta_0 \in \mathbf{M}$. Let $p \geq \frac{2n}{n+2}$, $\lambda \geq -\lambda_1$. Then for any $\eta \in C^1(\mathcal{I}; \mathbf{M})$ exists a solution of problem (4), (6), (7).

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