Martingales with respect to special filtrations

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The following simple model has been studied and intensively used in applications. Let Γ be a random time, i.e. a random variable with positive values on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Consider the smallest filtration with respect to which Γ is a stopping time (or, equivalently, the process $\mathbb{1}_{\{t\geqslant\Gamma\}}$ is adapted). Dellacherie [1] was the first who considered this model (in fact, with $\Omega = \mathbb{R}_+$ and $\Gamma(\omega) = \omega$; however, this specification is not important) to construct some counterexamples from the general theory of stochastic processes. His formula for the compensator of the process $\mathbb{1}_{\{t\geqslant\Gamma\}}$ is used for calculating compensators of marked point processes and in models of default risk.

Our aim is to study a more general new model introduced in [2]. Let Γ be a random variable with values in $\overline{\mathbb{R}}_+$ on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$. We tacitly assume that $\mathsf{P}(\Gamma > 0) > 0$. $G(t) = \mathsf{P}(\Gamma \leqslant t)$, $t \in \mathbb{R}_+$, stands for the distribution function of Γ and $\overline{G}(t) = 1 - G(t)$. Put also $t_G = \sup\{t \in \mathbb{R}_+ \colon G(t) < 1\}$ and $\mathcal{T} = \{t \in \mathbb{R}_+ \colon \mathsf{P}(\Gamma \geqslant t) > 0\}$. Note that $\mathsf{P}(\Gamma \notin \mathcal{T}) = 0$. We often distinguish between the following two cases:

Case A $\mathsf{P}(\Gamma = t_G < \infty) = 0$.

Case B $\mathsf{P}(\Gamma = t_G < \infty) > 0$.

It is clear that $\mathcal{T} = [0, t_G)$ in Case A and $\mathcal{T} = [0, t_G]$ in Case B.

According to [2], we define $\mathscr{F}_t$, $t \in \mathbb{R}_+$, as the collection of subsets A of Ω such that $A \in \mathscr{F}$ and $A \cap \{t < \Gamma\}$ is either $\varnothing$ or coincides with $\{t < \Gamma\}$. It is easy to check that $\mathscr{F}_t$ is a σ-field for every $t \in \mathbb{R}_+$. A random variable X is $\mathscr{F}_t$-measurable if and only if it is constant on $\{t < \Gamma\}$. Now it is trivial to check that $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is a filtration, i.e. an increasing right-continuous family. We call this filtration a *single jump filtration*. It is determined by generating elements Γ and $\mathscr{F}$. We will also write $\mathbb{F}(\Gamma, \mathscr{F})$ to distinguish it among other filtrations. The case where $\mathscr{F} = \sigma\{\Gamma\}$, i.e. $\mathscr{F}$ is the smallest σ-field with respect to which Γ is measurable will be referred to as the Dellacherie model.

Example 1. *Let Γ be as above, L integrable random variable. Since $\{\Gamma > t\}$ is an atom of $\mathscr{F}_t$ and the traces of $\mathscr{F}$ and $\mathscr{F}_t$ on $\{\Gamma \leqslant t\}$ coincide, then*

$$\mathsf{E}(L|\mathscr{F}_t) = F(t)\mathbb{1}_{\{t<\Gamma\}} + L\mathbb{1}_{\{t\geqslant\Gamma\}}, \quad t < t_G, \quad \textit{where} \quad F(t) = \overline{G}(t)^{-1}\int_{\{\Gamma>t\}} L\, d\mathsf{P}.$$

Every uniformly integrable martingale has this form. Note that $F(t)$, $t < t_G$, is a deterministic right-continuous function which is absolutely continuous with respect to $dG(t)$ on $[0, t]$ for every $t < t_G$. In particular, it has a finite variation over $[0, t]$ for $t < t_G$, and over $[0, t_G)$ in Case B.

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Example 2. *Let* Γ *be as above and correspond to Case* A. *Define*

$$M_t = \overline{G}(t)^{-1}\mathbb{1}_{\{t<\Gamma\}}. \tag{1}$$

Then $M = (M_t)_{t\in\mathbb{R}_+}$ *is a local martingale. Moreover, it is a martingale only if* $t_G = \infty$, *and it is a supermartingale if* $t_G < \infty$. *This process* (*if* $t_G = \infty$) *often appears in the modelling of credit risk.*

Example 3. *Let* $B = (B_t)_{t\in\mathbb{R}_+}$ *be a standard Brownian motion defined on some stochastic basis* $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\in\mathbb{R}_+}, \mathsf{P})$ *and* σ *an a.s. finite stopping time on this basis. Define*

$$\tau_t = \inf\{s \in \mathbb{R}_+ \colon B_s > t\} \quad \text{and} \quad X_t = B_{\sigma\wedge\tau_t}, \quad t \geqslant 0.$$

The process $X = (X_t)_{t\in\mathbb{R}_+}$ *can be written as*

$$X_t = t\mathbb{1}_{\{t<\Gamma\}} + B_\sigma \mathbb{1}_{\{t\geqslant\Gamma\}}, \quad \text{where} \quad \Gamma = \sup_{s\leqslant\sigma} B_s.$$

Define a single jump filtration $\mathbb{F}(\Gamma, \mathscr{G}) = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ *generated by* Γ. *In general,* X *is a submartingale with respect to* $(\mathscr{G}_{\tau_t})_{t\in\mathbb{R}_+}$ *and* $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, *see* [3], *and may not be a martingale, take, e.g.,* $\sigma = \tau_1$.

Let us mention briefly what is known about considered models with single jump filtrations. A simple characterisation of local martingales is suggested in [2]. As a consequence, a full description of all local martingales is given and they are classified according to their global behaviour. It is shown that there may be σ-martingales that are not local martingales [2] and there may be local martingales that are not local martingales with respect to the filtration that they generate [4]. Both possibilities are absent in the Dellacherie model, A full description of all σ-martingales is given [2] and it is shown that every σ-martingale remains to be a σ-martingale with respect to the filtration that it generates [4].

Here we consider a single jump filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ generated by a random time Γ and a σ-field $\mathscr{F}$ and characterize semimartingales with respect to it. Processes with finite variation are not assumed to be adapted and to start from 0. For a process $X = (X_t)_{t\in\mathbb{R}_+}$, X^Γ stands for the stopped process $(X_{t\wedge\Gamma})_{t\in\mathbb{R}_+}$.

Theorem 1. (i) *Every semimartingale is a process with finite variation.*

(ii) *If* X *is a semimartingale, the stopped process* X^Γ *has a representation*

$$X_t^\Gamma = F(t)\mathbb{1}_{\{t<\Gamma\}} + L\mathbb{1}_{\{t\geqslant\Gamma\}}, \quad t \in \mathbb{R}_+, \tag{2}$$

where $F\colon \mathbb{R}_+ \to \mathbb{R}$ *is a deterministic right-continuous function and* L *is a random variable. Moreover,* F *has a finite variation over* $[0,t]$ *for* $t < t_G$, *and over* $[0, t_G)$ *in Case* B. *Any process that satisfies* (2) *with* F *as described above, is a semimartingale.*

(iii) *If a process with finite variation vanishes on the stochastic interval* $[\![0, Gamma]\!]$, *then it is a predictable semimartingale.*

(iv) *Let* X *be a semimartingale. The following statements are equivalent*:

(a) X *is a special semimartingale.*
(b) $\mathsf{E}(|X_\Gamma|\mathbb{1}_{\{\Gamma\leqslant t\}}) < \infty$ *for all* $t \in \mathfrak{T}$.
(c) $\mathsf{E}(|\Delta X_\Gamma|\mathbb{1}_{\{\Gamma\leqslant t\}}) < \infty$ *for all* $t \in \mathfrak{T}$.

Remark 1. *Statement* (i) *is known for a class of so-called jumping filtrations, see* [5]. *A single jump filtration is a special case of a jumping filtration.*

Remark 2. *A formula for the canonical decomposition of a special semimartingale is a simple consequence of Theorem* 1 *and Theorem* 5 *in* [2].

Given a process $X = (X_t)_{t\geqslant 0}$, its running maximum process is denoted by $\overline{X} = (\overline{X}_t)_{t\geqslant 0}$:

$$\overline{X}_t := \sup_{s\leqslant t} X_s.$$

A process X is called max-continuous if the process $\overline{X}$ is continuous.

A max-continuous local martingale X, $X_0 = 0$, is called a local max-level martingale [3] if $\mathsf{E}(X_{C_t}) \equiv 0$, where $C_s := \inf\{t\colon \overline{X}_t > s\}$. A local max-level martingale is not necessary a martingale. Indeed, assume that Γ has a distribution with finite support and without atoms and define M by (1), then $M_t - M_0$ is a local max-level martingale but not a martingale.

Theorem 2. *A necessary and sufficient condition that a random vector* (W, V) *with values in* $[0, +\infty] \times [0, +\infty)$ *have the same joint law as the vector* $(\overline{X}_\infty, \overline{X}_\infty - X_\infty)$ *for some local max-level martingale* X *is that* $\{W = 0\} \cup \{W = \infty\} \subseteq \{V = 0\}$ *and*

$$\mathsf{E}(W \wedge t) = \mathsf{E}(V \mathbb{1}_{\{W\leqslant t\}}), \quad t \geqslant 0. \tag{3}$$

If the vector (W, V) *satisfies this condition, then one can take as* X *a stopped Brownian motion as in Example* 3 *which is a continuous martingale, or*

$$X_t = W \wedge t - V \mathbb{1}_{\{W\leqslant t\}}, \quad t \geqslant 0,$$

which is a martingale with respect to a single jump filtration generated by W, *or*

$$X_t = Q(t \wedge \Gamma) - V \mathbb{1}_{\{\Gamma\leqslant t\}}, \quad t \geqslant 0,$$

where $Q(t) := \inf\{s\colon \mathsf{P}(W \leqslant s) > t\}$ *is the upper quantile function of* W, Γ *has a uniform distribution on* $(0, 1)$ *and* $\mathrm{Law}(Q(\Gamma), V) = \mathrm{Law}(W, V)$. *In the latter case* X *is a local martingale with respect to a single jump filtration generated by* Γ.

Remark 3. *Let* X *be a local max-level martingale. Then* X_t *converges a.s. on the set* $\{\overline{X}_\infty < \infty\}$ *to a limit denoted by* X_∞. *We set* $\overline{X}_\infty - X_\infty = 0$ *on the set* $\{\overline{X}_\infty = \infty\}$.

Remark 4. *The quantile function* $Q(t)$ *is continuous if* W *satisfies* (3), *see* [3].

A non-trivial part of Theorem 2 is that under (3) there is a stopped Brownian motion $X = B^\sigma$ with $\mathrm{Law}(\overline{X}_\infty, \overline{X}_\infty - X_\infty) = (\overline{B}_\sigma, \overline{B}_\sigma - B_\sigma) = \mathrm{Law}(W, V)$, see [3].

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